Chapter 4

Pole placement

4.1 Control synthesis in state space

Generally, if a state space representation is available, either in continuous or in discrete time, the controller design problem is many times formulated as *finding a control law of the form* $\boldsymbol{u} = K(\cdot)\boldsymbol{x}$ such that the closed-loop system is 1) stable and 2) satisfies some performance measures, such as convergence rate, robustness, disturbance attenuation, etc. Naturally, a prerequisite for controller design is that the system is controllable.

If all the states are available (measured) at all times, then a full state feedback – i.e., using all information related to the states – can be used. If this is not the case, either an output feedback controller or an observer – that estimates the states and the estimates can be used for control purposes – is needed. In this course we consider the latter.

For the case when an observer is used together with the controller, the control loop for linear systems is shown in Figure 4.1. Note that the same loop is valid for discrete-time systems.

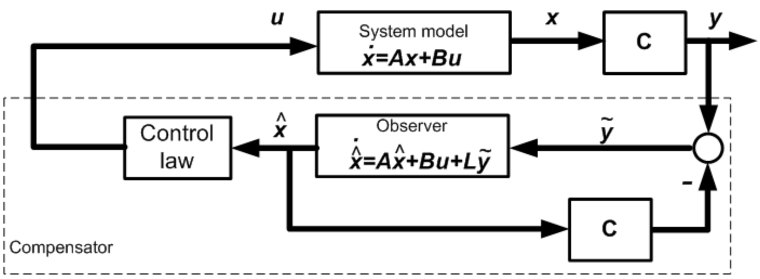

Figure 4.1: State variable compensator with observer

In what follows, we first present state-feedback controller design and then observer design for LTI systems.

4.2 Pole assignment by state feedback

We shall present a design method commonly called *pole placement*. We assume that all state variables are measurable and available for feedback. If the system is completely state controllable, i.e., the controllability matrix has full row rank, the poles of the closed-loop system may be placed at any desired location by means of state feedback through an appropriate state feedback matrix.

The purpose of the control law is to allow us to assign a set of pole locations for the closed-loop system that will correspond to satisfactory dynamic response in terms of transient response specifications.

The first step in pole-placement is the selection of the pole locations of the closed-loop system. It is always useful to keep in mind that the control effort required in related to how far the open-loop poles are moved by the feedback. Furthermore, when a zero is near a pole, the system may be nearly uncontrollable and moving such poles requires large control gains and a large control effort.

Recall that the overshoot, rise time, peak time and settling time can be related directly to the poles location. The closed-loop poles for a higher order system can be chosen as a desired pair of dominant second-order poles and the rest of the poles selected such that they have real parts corresponding to sufficiently damped exponential terms in the system response. In such a case, the closed-loop system will mimic a second-order response. We also must make sure that the zeros are far enough into the left half-plane to avoid any appreciable effect on the second-order behavior, (Franklin et al., 2006).

4.2.1 Full-state feedback control law by pole placement

Consider a system described in state-space by:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned} \tag{4.1}$$

where $\boldsymbol{x}$ is the $n \times 1$ state vector, $\boldsymbol{u}$ is the input (control) vector, $\boldsymbol{y}$ the output vector, A is an $n_x \times n_x$ matrix (the system matrix), B is an $n_x \times n_u$ input matrix and C is an $n_y \times n_x$ output matrix.

All states are assumed available for feedback and the system is controllable.

The control law will be designed such that all state variables will approach zero in steady-state from non-zero initial conditions caused for example by external disturbances.

We choose the control law such that each control input is a linear combination of the state variables:

$$\begin{aligned}
\boldsymbol{u} &= -K\boldsymbol{x} \\
&= -\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n_x} \\ k_{21} & k_{22} & \cdots & k_{2n_x} \\ \vdots & \vdots & \vdots & \vdots \\ k_{n_u1} & k_{n_u2} & \cdots & k_{n_un_x} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_{n_x} \end{pmatrix} \\
&= -\begin{pmatrix} k_{11}x_1 + k_{12}x_2 + \cdots + k_{1n_x}x_{n_x} \\ k_{21}x_1 + k_{22}x_2 + \cdots + k_{2n_x}x_{n_x} \\ \vdots \\ k_{n_u1}x_1 + k_{n_u2}x_2 + \cdots + k_{n_un_x}x_{n_x} \end{pmatrix}
\end{aligned}$$

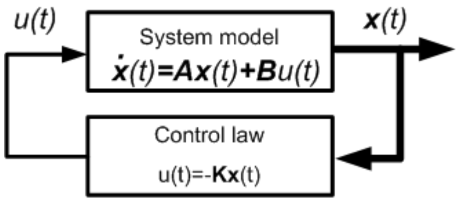

Figure 4.2: State feedback control law

As shown in Figure 4.2 and as resulted from equation (4.2.1) the system has a constant matrix in state feedback path.

Substituting the feedback law given by (4.2.1) into the state equation of system (4.1) we obtain the state equation for the closed-loop system:

$$\dot{\boldsymbol{x}} = Ax - BK\boldsymbol{x} = (A - BK)\boldsymbol{x} \tag{4.2}$$

The stability of the closed-loop system (4.2) and its transient characteristics are determined by the eigenvalues of the closed-loop system matrix $A - BK$. If the system is controllable, the poles of the closed-loop system can be placed in any desired location by the appropriate choice of the matrix K.

The characteristic equation of the closed-loop system is:

$$\det[sI - (A - BK)] = 0 \tag{4.3}$$

When evaluated, this gives an n-th order polynomial in s containing the elements of the gain matrix K: k_{11}, k_{12}, ..., $k_{n_un_x}$. The control law design then consists of computing the gain K so that the roots of equation (4.3) are in desirable locations.

Assume that the desired location of the closed-loop poles are known:

$$\text{desired poles: } p_1,\ p_2,\ \cdots,\ p_n$$

The corresponding desired characteristic equation is:

$$(s - p_1)(s - p_2) \cdots (s - p_n) = 0 \tag{4.4}$$

The required elements of K are obtained by matching coefficients of equations (4.3) and (4.4). This will result in the closed-loop system characteristic equation to be identical to the desired one and the poles to be placed in the desired location.

Example 4.1 *Consider the continuous-time dynamic system*

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + b\boldsymbol{u}\\ \boldsymbol{y} &= I\boldsymbol{x}\end{aligned}$$

i.e., all states are measured and available for feedback, with matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. *This system is unstable, but controllable – already rank*$(B) = 2$ *– thus we design a state-feedback control law* $\boldsymbol{u} = -K\boldsymbol{x}$ *to stabilize it. Let us impose the closed-loop system poles* $p_1 = -1$ *and* $p_2 = -2$. *With these poles, the characteristic equation of the closed-loop system is* $(s+1)(s+2) = 0$, *i.e.,* $s^2 + 3s + 2 = 0$.

Now we compute the characteristic equation as a function of the controller gains. The closed-loop system dynamics are given by

$$\dot{\boldsymbol{x}} = (A - BK)\boldsymbol{x}$$

where $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$ *and the characteristic equation is*

$$\det(sI - (A - BK)) = 0$$

i.e.,

$$\det \begin{pmatrix} s - 1 + k_{11} & 2k_{12} - 2 \\ k_{21} - 3 & s - 4 + 2k_{22} \end{pmatrix} = 0$$

resulting in

$$\begin{aligned}&(s - 1 + k_{11})(s - 4 + 2k_{22}) - (2k_{12} - 2)(k_{21} - 3) = 0\\ &s^2 + s(-5 + 2k_{22} + k_{11}) + (-2 - 2k_{22} - 4k_{11} - 2k_{11}k_{22} - 2k_{12}k_{21} + 2k_{21} + 6k_{12}) = 0\end{aligned}$$

The controller gains are computed by matching the coefficients of s:

$$\begin{aligned}&-5 + 2k_{22} + k_{11} = 3\\ &-2 - 2k_{22} - 4k_{11} - 2k_{11}k_{22} - 2k_{12}k_{21} + 2k_{21} + 6k_{12} = 2\end{aligned}$$

There are two equations and four unknowns, and an infinite number of solutions for k_{11}, k_{12}, k_{21}, *and* k_{22}. *For instance, two solutions are:* $K = \begin{pmatrix} 12 & 0 \\ 0 & -2 \end{pmatrix}$, $K = \begin{pmatrix} -1 & 0 \\ 0 & \frac{9}{2} \end{pmatrix}$, *both leading to the same closed-loop poles* 1 *and* 2.

todo: simulations

As we have seen in the above example, in the general case there are a number of solutions, but matching the coefficients of the characteristic equations may lead to equations that are quite cumbersome. A way to solve the pole placement problem is as follows.

- Let Λ a real matrix having the desired eigenvalues. For instance, Λ can be chosen as

$$\Lambda = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots \\ -\beta_1 & \alpha_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- If the system is controllable, then the closed-loop system poles can be placed at the desired locations. This means that there exists a transformation matrix T so that $T^{-1}(A - BK)T = \Lambda$, i.e., $AT - BKT = \Lambda T$.

- This equation is solved by splitting it as follows:

$$\begin{aligned} AT - T\Lambda &= BG \\ KT &= G \end{aligned} \tag{4.5}$$

  The first equation in (4.5) is called the Sylvester equation and is linear in T. If G is chosen arbitrarily, the Sylvester equation can be solved in T.

- The feedback gains are recovered as $K = GT^{-1}$, where T is the solution obtained at the previous item.

Note that the procedure above gives a solution if A and Λ have no common eigenvalues, however, it may fail due to numerical problems (T not invertible or poorly conditioned) for special choices of G.

Example 4.2 ***todo+ discrete***

4.2.2 Pole placement for SISO systems

Let us now consider the case when the system considered is single input and single outputs, i.e., the input matrix B is a column vector of dimensions $n_x \times 1$. While the methods presented in the previous section – direct computation or solving the Sylvester equation – are applicable for SISO systems, we consider here the special case of transformation into controller canonical form. We have already seen in Chapter 3 that if the system is completely controllable, it can be transformed into the controller canonical form (repeated here for convenience):

$$A = \begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $\det(sI-A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is the characteristic polynomial of A.

Looking at the controller gain, the matrix K is a constant $1 \times n$ row vector, i.e., $K = \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix}$ and the state-feedback can be computed as represented in Figure 4.3. If the system is in controller canonical form, the closed-loop system can be written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= (A - BK)\boldsymbol{x} \\ &= \left(\begin{pmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} k_1 & k_2 & \cdots & k_n \end{pmatrix} \right) \boldsymbol{x} \end{aligned}$$

resulting in

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -(a_{n-1}+k_1) & -(a_{n-2}+k_2) & \cdots & -(a_1+k_{n-1}) & -(a_0+k_n) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \boldsymbol{x} \tag{4.6}$$

Looking at the form of the closed-loop system matrix in (4.6) it is quite clear that the closed-loop system poles can be chosen arbitrarily.

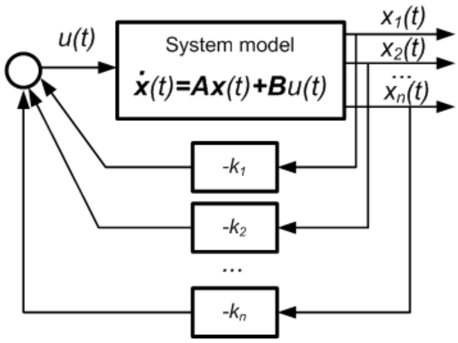

Figure 4.3: State feedback control law – SISO case

Example 4.3 *, (Ogata, 2002) Consider the system described by the state equation:*

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

We will design a state feedback control law $u = -\mathbf{Kx}$ such that the closed-loop poles are located at:

$$p_1 = -2 + j4, \quad p_2 = -2 - j4; \quad p_3 = -10$$

First we have to check the controllability matrix of the open-loop system. The controllability matrix is:

$$\mathbf{P_C} = \begin{bmatrix} B & \mathbf{AB} & \mathbf{A^2B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

and we find the determinant is $\det \mathbf{P_C} = -1$. Since the rank of the controllability matrix is $rank\ \mathbf{P_C} = 3$, the system is controllable.

If we consider that the output of this system is the first state variable $y = x_1$, the system response for a zero input $u = 0$ and initial conditions $\boldsymbol{x}(0) = [1\ 0\ 0]^T$ is shown in Figure 4.4.

The open-loop system is stable and the settling time of the open-loop response is about 14 seconds. Indeed, if we compute the eigenvalues of matrix A from:

$$\det(\lambda I - A) = 0, \text{ the eigenvalues are: } \lambda_1 = -5.05, \quad \lambda_2 = -0.31, \quad \lambda_3 = -0.64$$

The eigenvalues of the system matrix are also the system poles. As they have real and negative values the open-loop system is stable. We may want to improve the settling time of the system, we will compute the control law that moves the closed-loop poles

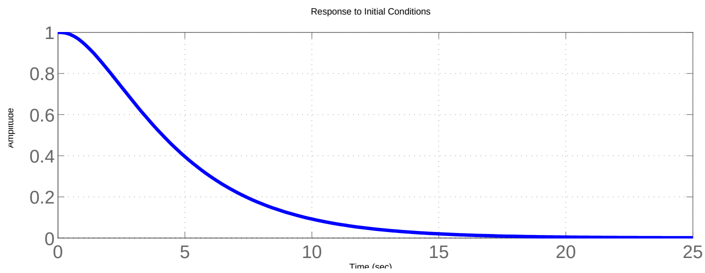

Figure 4.4: Response of the open-loop system to initial conditions

in the locations given by p_1, p_2 and p_3. The dominant desired poles are $p_{1,2}$ with the real part -2. The settling time of this pair of poles can be approximated as $t_s = 4/2 = 2sec$.

We apply now the pole placement method for computing the gain matrix $\mathbf{K}$:

- *For a third order system with one input and one output, the gain matrix is:*

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

and the characteristic equation of the closed-loop system, given by equation (4.3) is:

$$\det[sI - (\mathbf{A} - \mathbf{BK})] = 0$$

or

$$\det\left(\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{bmatrix} = s^3+(6+k_3)s^2+(5+k_2)s+1+k_1 = 0 \tag{4.7}$$

- *The characteristic equation of the closed-loop system for the desired poles is:*

$$(s-p_1)(s-p_2)(s-p_3) = (s+2-j4)(s+2+j4)(s+10) = s^3+14s^2+60s+200 = 0 \tag{4.8}$$

- *If we set the characteristic equation of the closed-loop system (4.7) equal to the characteristic equation for the desired poles (4.8) we obtain:*

$$6+k_3 = 14, \quad 5+k_2 = 60, \quad 1+k_1 = 200$$

from which we obtain:

$$k_1 = 199, \quad k_2 = 55, \quad k_3 = 8$$

or, the feedback gain matrix is:

$$\mathbf{K} = \begin{bmatrix} 199 & 55 & 8 \end{bmatrix}$$

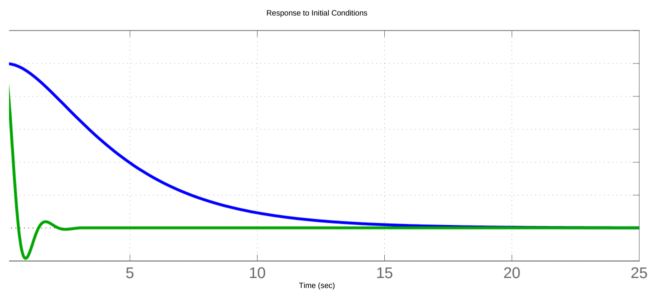

Figure 4.5: Response of the closed-loop system to initial conditions (green line) compared to the response of the open-loop system (blue line)

The settling time is indeed about 2 seconds, although the response of the closed-loop system exhibits some oscillations

4.2.3 Tracking systems – SISO case

Consider the system described by the state-space model:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\boldsymbol{x} + Bu \\ y = C\boldsymbol{x} \end{cases} \tag{4.9}$$

where $\boldsymbol{x}$ is the $n \times 1$ state vector, u is the (scalar) control signal, y the (scalar) output signal, A is an $n \times n$ scalar matrix (the system matrix), B is an $n \times 1$ constant matrix (the input matrix) and C is an $1 \times n$ constant matrix (the output matrix).

The goal now, is to drive the output y to a given reference input r, with zero-steady-state error.

One solution is to scale the reference input r and choose the control law (control signal) to be a linear combination of the state variables (see Figure 4.6):

$$u = Nr - \mathbf{K}\boldsymbol{x} = Nr - \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = Nr - k_1x_1 - k_2x_2 - \cdots - k_nx_n \tag{4.10}$$

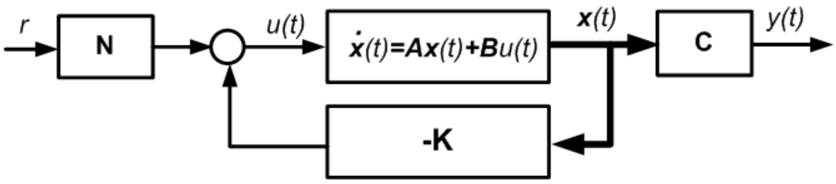

Figure 4.6: State feedback for tracking a reference input

where r and N are scalar values for a SISO system.

The closed-loop system is then:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \mathbf{A}\mathbf{x} + B(Nr - \mathbf{K}\boldsymbol{x}) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + BNr \\ y &= Cx\end{aligned}$$

The feedback matrix $\mathbf{K}$ is determined by pole placement so that the closed-loop system has a characteristic equation given by a set of desired closed-loop poles.

At steady-state the derivative of the state vector is zero ($\dot{\boldsymbol{x}} = 0$). If the steady-state value of the state vector is $\boldsymbol{x}_{ss}$ and the steady-state value of the output is y_{ss}, we have:

$$\begin{aligned}0 &= (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_{ss} + BNr \\ y_{ss} &= Cx_{ss}\end{aligned}$$

or

$$\begin{aligned}(\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_{ss} &= -BNr, \quad \Rightarrow \boldsymbol{x}_{ss} = -(\mathbf{A} - \mathbf{B}\mathbf{K})^{-\mathbf{1}}\mathbf{B}Nr \\ y_{ss} &= Cx_{ss} = -C(\mathbf{A} - \mathbf{B}\mathbf{K})^{-\mathbf{1}}\mathbf{B}Nr\end{aligned}$$

The steady-state error is zero if $y_{ss} = r$, so the value of N can be computed from:

$$y_{ss} = -C(\mathbf{A} - \mathbf{B}\mathbf{K})^{-\mathbf{1}}\mathbf{B}Nr = r, \quad \Rightarrow \quad N = -\frac{1}{C(\mathbf{A} - \mathbf{B}\mathbf{K})^{-\mathbf{1}}\mathbf{B}}$$

Example 4.4 *Consider the system described in state-space by:*

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} \boldsymbol{x}\end{aligned}$$

We will design a control system using state-feedback so that the closed-loop system has two poles located at $p_{1,2} = -1 \pm j$ and the steady-state error for a step input is zero.

- *We check controllability first. The controllability matrix, the determinant and the rank are:*

$$\mathbf{P_C} = \begin{bmatrix} B & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \ \det \mathbf{P_C} = 1, \ \Rightarrow \ \text{rank}\mathbf{P_C} = 2$$

The system is controllable.

- *The feedback gain matrix $\mathbf{K}$ is obtained by pole placement. If the desired poles of the closed-loop system are $p_{1,2} = -1 \pm j$, the characteristic equation for these poles is:*

$$(s + 1 - j)(s + 1 + j) = s^2 + 2s + 2 = 0$$

- *The characteristic equation of the closed-loop system with the feedback gain $\mathbf{K} = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is:*

$$\begin{aligned} \det[sI - (\mathbf{A} - \mathbf{BK})] &= \det \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) = \\ &= \begin{bmatrix} s - 2 + k_1 & -3 + k2 \\ -1 & s \end{bmatrix} = 0 \end{aligned}$$

or:

$$s(s - 2 + k_1) - (-1)(-3 + k_2) = s^2 + (-2 + k_1)s - 3 + k_2 = 0$$

- *By setting the characteristic equation of the closed loop system equal to the characteristic equation for the desired poles we obtain:*

$$-2 + k_1 = 2, \quad -3 + k_2 = 2, \quad \text{and} \quad k_1 = 4, \quad k_2 = 5$$

and the feedback gain matrix is:

$$\mathbf{K} = \begin{bmatrix} 4 & 5 \end{bmatrix}$$

Note that this matrix will only ensure the desire transient characteristics (according to the desired closed-loop poles). The zero steady-state error for a step input is not guaranteed. Therefore, we will compute the gain N that will scale the reference input so that the output equals the reference at steady-state.

- *Compute N from:*

$$N = -\frac{1}{C(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}}$$

$$(\mathbf{A} - \mathbf{BK}) = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}$$

$$(\mathbf{A}-\mathbf{BK})^{-1} = \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \end{bmatrix}$$

$$C(\mathbf{A}-\mathbf{BK})^{-\mathbf{1}}\mathbf{B} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1/2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1/2$$

$$N = -\frac{1}{1/2} = -2$$

The step response of the closed-loop system, implemented according to Figure 4.6, is shown in Figure 4.7.

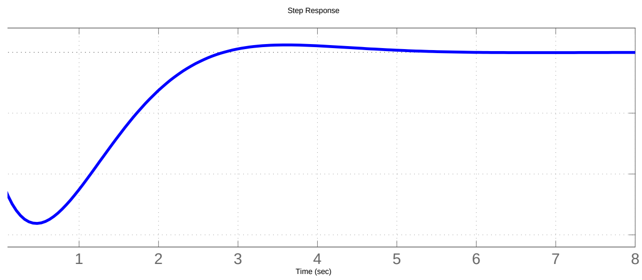

Figure 4.7: Step response of the closed-loop system

The output follows the unit step, thus $y_{ss} = 1$ *and the steady-state error is zero:* $e_{ss} = 0$.

4.2.4 Full-order observer design by pole placement

Let us consider now a system described in state-space by:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned} \tag{4.11}$$

where $\boldsymbol{x}$ is the $n \times 1$ state vector, $\boldsymbol{u}$ is the input (control) vector, $\boldsymbol{y}$ the output vector, A is an $n_x \times n_x$ matrix (the system matrix), B is an $n_x \times n_u$ input matrix and C is an $n_y \times n_x$ output matrix.

As it is frequently the case, not all states can be measured, and only the measurements $\boldsymbol{y}$ are available. We therefore introduce an *observer*, which is a mechanism to estimate the unmeasured states from the available measurements. An observer is in fact a *virtual sensor*.

A general linear observer has the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= A\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}} \end{aligned} \tag{4.12}$$

where $\widehat{\boldsymbol{x}}$ is the $n \times 1$ state vector denoting the estimated states, $\widehat{\boldsymbol{y}}$ is the estimated output, and L is the observer gain to be determined. Note that for $L = 0$, the observer (4.12) degenerates into the system (4.11). The term $L(\boldsymbol{y} - \widehat{\boldsymbol{y}})$ is called the *innovation process* and represents the feedback error between the observation and the predicted model output.

The general goal of observer design is to make the *estimation error* defined as the difference between the true and estimated stated $\boldsymbol{e} = \boldsymbol{x} - \widehat{\boldsymbol{x}}$ converge to zero.

The estimation error dynamics, i.e., $\dot{\boldsymbol{e}}$ can be written as

$$\begin{aligned}\dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}} - \dot{\widehat{\boldsymbol{x}}} \\ &= A\boldsymbol{x} + B\boldsymbol{u} - (A\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\end{aligned}$$

i.e,

$$\dot{\boldsymbol{e}} = (A - LC)\,\boldsymbol{e} \tag{4.13}$$

Remark: As can be seen, the input $\boldsymbol{u}$ – as long as it known and available independently of the estimated states – does not affect the observer design.

If the system is completely observable, it is possible to assign the eigenvalues of $(A - LC)$ *to any desired location.*

The actual assignment is similar to the pole placement in the state feedback case.

The characteristic equation of the error dynamics is:

$$\det[sI - (A - LC)] = 0 \tag{4.14}$$

which is an n-th order polynomial in s containing the elements of the gain matrix L: l_{11}, l_{12}, ..., $l_{n_x n_y}$. The observer design then consists of computing the gain L so that the roots of equation (4.14) are in desirable locations.

Assume that the desired location of the closed-loop poles are known:

$$\text{desired poles: } p_1,\ p_2,\ \cdots,\ p_n$$

The corresponding desired characteristic equation is:

$$(s - p_1)(s - p_2)\cdots(s - p_n) = 0 \tag{4.15}$$

The elements of L are obtained by matching coefficients of equations (4.14) and (4.15) leading to the error dynamics characteristic equation to be identical to the desired one.

Example 4.5 *Consider the continuous-time dynamic system*

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} \\ \boldsymbol{y} &= C\boldsymbol{x}\end{aligned}$$

with $A = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$ *and* $C = \begin{pmatrix} 1 & 1 \end{pmatrix}$*, i.e., the states are not measured. Our objective is to estimated the two states. The observability matrix is* $\Gamma_o = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$*, with* $\mathrm{rank}(\Gamma_o) = 2$*, thus the system is completely observable and an observer can be designed.*

Let us impose the error dynamics poles $p_1 = -5$ *and* $p_2 = -6$*. With these poles, the characteristic equation of the closed-loop system is* $(s+5)(s+6) = 0$*, i.e.,* $s^2 + 11s + 30 = 0$*.*

Now we compute the characteristic equation as a function of the observer gains. The error dynamics are given by

$$\dot{\hat{\boldsymbol{x}}} = (A - LC)\boldsymbol{e}$$

where $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ *and the characteristic equation is*

$$\det(sI - (A - Lc)) = 0$$

i.e.,

$$\det \begin{pmatrix} s - 1 + l_1 & l_1 - 1 \\ l_2 - 3 & s - 2 + l_2 \end{pmatrix} = 0$$

resulting in

$$s^2 + s(-3 + l_1 + l_2) + (l_1 - 1) = 0$$

The observer gains are computed by matching the coefficients of s*:*

$$\begin{aligned} -3 + l_1 + l_2 &= 11 \\ l_1 - 1 &= 30 \end{aligned}$$

and result in $l_1 = 31$ *and* $l_2 = -17$*.*

todo: simulations

Remark: Note that the system in Example 4.5 is not stable. However, this fact does not affect the design of the observer.

Similarly to controller design, in the general case there are a number of solutions. A way to solve the pole placement problem is as follows.

- Let Λ a real matrix having the desired eigenvalues. For instance, Λ can be chosen as

$$\Lambda = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & \cdots \\ -\beta_1 & \alpha_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \lambda_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

- If the system is observable, then the poles of the error dynamics can be placed at the desired locations. This means that there exists a transformation matrix T so that $T(A - LC)T^{-1} = \Lambda$, i.e., $TA - TLC = \Lambda T$.
- This equation is solved by splitting it as follows:

$$\begin{aligned} TA - \Lambda T &= GC \\ TL &= G \end{aligned} \tag{4.16}$$

- The observer gains are recovered as $L = T^{-1}G$, where T is the solution obtained at the previous item.

Note that the procedure above is very similar to that used in controller design. In fact, by taking the transpose of the two equations in (4.16), we have

$$\begin{aligned} A^T T^T - T^T \Lambda^T &= C^T G^T \\ L^T T^T &= G^T \end{aligned}$$

Comparing it to (4.5), we see that A^T corresponds to A, C^T to B, L^T to K, etc., and is a direct consequence of *duality*.

Example 4.6 ***todo+ discrete***

4.2.5 SISO systems in observer canonical form

Let us now consider the case when the system considered is single input and single output and is in observer canonical form. We have already seen in Chapter 3 that if the system is completely observable, it can be transformed into the observer canonical form (repeated here for convenience):

$$A = \begin{pmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ is the characteristic polynomial of A.

In this case, the observer gain L is a constant $n_x \times 1$ column vector, i.e., $L = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n_x} \end{pmatrix}$. Due to the system being in observer canonical form, the error dynamics can be written as

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= (A - LC)\boldsymbol{e} \\
&= \left(\begin{pmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0 & 0 & \cdots & 0 & 0 \end{pmatrix} - \begin{pmatrix} l_1 \\ l_2 & \vdots & l_{n_x} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \vdots & 0 \end{pmatrix} \right) \boldsymbol{e}
\end{aligned}
$$

resulting in

$$
\dot{\boldsymbol{e}} = \begin{pmatrix} -a_{n-1} - l_1 & 1 & \cdots & 0 & 0 \\ -a_{n-2} - l_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 - l_{n-1} & 0 & \cdots & 0 & 1 \\ -a_0 - l_n & 0 & \cdots & 0 & 0 \end{pmatrix} \boldsymbol{e} \tag{4.17}
$$

and it is quite clear that the poles of the error dynamics can be chosen arbitrarily.

Example 4.7 ***todo*** *Consider the system with* $A = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$

4.3 Separation principle

Finally, let us consider the case when a control law is required – for instance to stabilize the system –, but not all states are measured, i.e, the system is of the general form

$$
\begin{aligned}
\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\
\boldsymbol{y} &= C\boldsymbol{x}
\end{aligned} \tag{4.18}
$$

and both an observer and a controller have to be designed. It is assumed that the system is completely controllable and completely observable. The observer will have the form (4.12), repeated here for convenience

$$
\begin{aligned}
\dot{\widehat{\boldsymbol{x}}} &= A\widehat{\boldsymbol{x}} + B\boldsymbol{u} + L(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \\
\widehat{\boldsymbol{y}} &= C\widehat{\boldsymbol{x}}
\end{aligned}
$$

while the controller considered is – since the true states are not available –

$$
\boldsymbol{u} = -K\widehat{\boldsymbol{x}} \tag{4.19}
$$

The error dynamics are – note that the input $\boldsymbol{u}$ applied is the same for the system and the observer –

$$
\dot{\boldsymbol{e}} = (A - LC)\boldsymbol{e} \tag{4.20}
$$

and the closed-loop system dynamics are:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ &= A\boldsymbol{x} - BK\widehat{\boldsymbol{x}} \\ &= A\boldsymbol{x} - BK(\boldsymbol{x} - \boldsymbol{e}) \\ &= (A - BK)\boldsymbol{x} + BK\boldsymbol{e} \end{aligned} \tag{4.21}$$

Taking the two dynamics together, we have

$$\begin{pmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{e}} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{e} \end{pmatrix} \tag{4.22}$$

The characteristic equation of (4.22) – since the system matrix is upper-block triangular – will be given by multiplications of the characteristic equations of $A - BK$ and $A - LC$. Thus, as long as both the error dynamics and the state-feedback dynamics $A - BK$ are stable, the observer-based state-feedback closed-loop will be stable. This fact – i.e., that the observer and the controller can be designed independently – is called the *separation principle*.

Note however, that although stability is maintained in the absence of perturbations, model uncertainty, etc., the performances of the closed-loop system will depend on the whole system dynamics.

Remark: From an application point of view, when placing the poles, the error dynamics should be at least $5 - 6$ times faster than the state-feedback closed-loop dynamics.