# Observer design for time-delay TS fuzzy systems with nonlinear consequents

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**Abstract:** In this paper we consider an observer design method for time-delay Takagi-Sugeno fuzzy models with nonlinear consequents. We assume that both the input and the membership functions are affected by the known delay. The nonlinearities in the consequents are handled by a slope-bounded condition. The observer design conditions are formulated as linear matrix inequalities. A numerical example illustrates the results.

*Keywords:* TS fuzzy systems, time-delay, observer design, Lyapunov synthesis, LMIs

## 1. INTRODUCTION

State estimation is an important problem for real systems. Where direct measurements are not physically possible or where the sensors are too expensive, a state observer can be used that estimates the unmeasured states from the available measurements, e.g. (Bergsten et al., 2002; Guerra et al., 2017; Ichalal et al., 2018, 2010; Quintana et al., 2020).

The dynamic model of a system is usually nonlinear. A popular method for handling nonlinearities is the Takagi-Sugeno (TS) fuzzy modelling. TS models are a convex combination of local, usually linear models. A disadvantage of the fuzzy modelling is that the number of rules can be prohibitive. In order to reduce the number of local models, we consider a form where each local model may be nonlinear. Furthermore, if some nonlinearities depend on unmeasured states, they can also be included in the nonlinear local models, thus we separate the measured and unmeasured-state nonlinearities. In what follows, we will refer to such fuzzy models as models with nonlinear consequents.

In the literature nonlinear consequents are usually handled using a Lipschitz condition (Mazenc et al., 2012; Nguyen et al., 2017; Van Assche et al., 2011; Zemouche and Boutayeb, 2011). A less conservative, one-sided Lipschitz and inner-bounded condition is used by Nguyen et al. (2021). In this paper we use a different approach, by assuming that the nonlinearities are slope-bounded instead of sector bounded, inspired by the work of Arcak and Kokotović (1999).

Time-delay is a frequent research subject. It appears in many applications, where the sensors and actuators are not co-located, see e.g., Chang and Chen (2010); Laurain et al. (2017); Ma et al. (2013); Mazenc et al. (2012), etc. Time-delays are frequently non-negligible factors, thus it is important to take them into account.

Nonlinear systems with input delays, where the delay is assumed to be differentiable, were studied by Léchappé et al. (2018). The problem with the differential conditions on delays is that they are difficult to verify as the delays are unknown. Moreover, the unknown delays may present jumps or discontinuities (Bresch-Pietri et al., 2018).

When designing controllers it is generally assumed that the maximum delay and the maximum of the derivative are known (Lian et al., 2020; Lin et al., 2006; Mátyás et al., 2020). On the other hand, when designing observers, it is generally assumed that the delay is known (Nguyen et al., 2017; Van Assche et al., 2011).

Motivated by the aforementioned works, we propose an observer design for delayed TS fuzzy system with nonlinear consequents, where the membership functions may depend on both current and delayed states and where the delay and derivative of the delay are known. In our previous work (Mátyás et al., 2020) we considered a controller design method on the same system.

The paper is structured as follows: in Section II we introduce the necessary background, assumptions and the problem statement. Sufficient conditions for observer design are presented in Section III. In Section IV we illustrate the developed conditions on a numerical example and compare them with another example from the literature. Section V concludes the paper.

**Notations**. Let $F = F^T \in R^{n\times n}$ be a real symmetric matrix; $F > 0$ and $F < 0$ mean that $F$ is positive definite and negative definite, respectively. $I$ denotes the identity matrix and 0 the zero matrix of appropriate dimensions. The symbol $*$ in a matrix indicates a transposed quantity in the symmetric position, for instance $\begin{pmatrix} P & * \\ A & P \end{pmatrix} = \begin{pmatrix} P & A^T \\ A & P \end{pmatrix}$, and $A + * = A + A^T$. The notation $\text{diag}(f_1, ..., f_n)$, where $f_1, ..., f_n \in \mathbb{R}$, stands for the diagonal matrix, whose diagonal components are $f_1, ..., f_n$. $\|x\|$, where $x \in \mathbb{R}^{n_x}$, is the Euclidean norm of $x$. Throughout this paper, the following shorthand notations are used to represent convex sums of matrix expressions:

$$F_z = \sum_{i=1}^{s} q_i(z(t))F_i, \tag{1}$$

$$F_{z_\tau} = \sum_{i=1}^{s} q_i(z(t-\tau(t)))F_i, \quad (2)$$

$$F_{zz_\tau} = \sum_{i=1}^{s} q_i(z(t)) \sum_{j=1}^{s} q_j(z(t-\tau(t)))F_{ij}, \quad (3)$$

where $q_i$, $i = 1, ..., s$ are nonlinear functions called the membership functions with the property:

$$q_i \in [0,\, 1],\, i = 1, ..., s, \quad \sum_{i=1}^{s} q_i(z) = 1. \quad (4)$$

## 2. PRELIMINARIES AND PROBLEM STATEMENT

The time-delay TS fuzzy model that we consider has the following special structure:

$$\begin{aligned} \dot{x}(t) =& A_{zz_\tau}x(t) + D_{zz_\tau}x(t-\tau(t)) + G_{zz_\tau}\psi(Hx(t)), \\ x(t) =& \phi(t), \quad t \leq \tau(t), \\ y(t) =& Cx(t), \end{aligned} \quad (5)$$

where $A_{zz_\tau} \in \mathbb{R}^{n_x \times n_x}$, $D_{zz_\tau} \in \mathbb{R}^{n_x \times n_x}$, $G_{zz_\tau} \in \mathbb{R}^{n_u \times r}$ and $C \in \mathbb{R}^{n_y \times n_x}$ represent the model matrices, i.e., $\sum_{i=1}^{s}\sum_{j=1}^{s} q_i(z(t))q_j(z(t-\tau(t)))A_{ij}x(t)$, etc.; $x(t) \in \mathbb{R}^{n_x}$ is the state vector, $y(t) \in \mathbb{R}^{n_y}$ is the output vector, $s$ is the number of rules, $z(t) \in \mathbb{R}^{n_z}$ is the premise vector; $\tau(t)$ is the varying time-delay, where $\tau(t)$ is differentiable, $\dot{\tau}(t) \leq d$, $d \in [0,\, 1)$ is a given constant, $\tau(t) \leq h$, $h > 0$ is the maximum time-delay; $\phi(t)$ is the initial condition of the state defined on $0 \leq t \leq \tau(t)$. As can be seen in (5) the local models have a nonlinear term, therefore in what follows we will refer to (5) as a fuzzy model with nonlinear consequents. In what follows for simplicity we omit in the notation the explicit time dependence of the delay, i.e., we use $\tau$ instead of $\tau(t)$, $\dot{\tau}$ instead of $\dot{\tau}(t)$.

*Assumption 1.* We consider that the premise vector $z(t)$ is measured, the delayed premise vector $z(t-\tau)$ is available and the varying time-delay $\tau$ is also known.

Note that such assumptions, although unrealistic, are frequently encountered, see e.g., Van Assche et al. (2011). $\psi(Hx(t)) \in \mathbb{R}^r$ is an $r$-dimensional vector where $H \in \mathbb{R}^{r \times n_x}$ and each entry is a function of a linear combination of the states, i.e.,

$$\psi_i = \psi_i(\sum_{j=1}^{n} H_{ij}x_j), \quad i = 1, ..., r.$$

The term $\psi(Hx(t))$ includes all the nonlinearities that are not available from the measurements. This is a major issue for the observer design via convex structures. We assume that $\psi(Hx(t))$ satisfies:

*Assumption 2.* For any $i \in \{1, ..., r\}$ there exist constants $0 < b_i \leq \infty$, so that

$$0 \leq \frac{\psi_i(v) - \psi_i(w)}{v - w} \leq b_i, \quad \forall v, w \in \mathbb{R}, v \neq w. \quad (6)$$

Note that if the expression (6) is lower bounded not by 0, but a constant, it can still be transformed to satisfy Assumption 2.

Assumption 2 intuitively bounds the rate of change of the nonlinearity, and corresponds to a global Lipschitz property of $\psi_i$, when $\psi_i$ is continuously differentiable. This assumption is made by Arcak and Kokotović (1999, 2001); Chong et al. (2012); Draa et al. (2018).

This means that there exist $\delta_i(t) \in [0,\, b_i]$, so that for any $v,\, w \in \mathbb{R}$

$$\psi_i(v) - \psi_i(w) = \delta_i(t)(v - w). \quad (7)$$

We denote $\delta(t) = \mathrm{diag}(\delta_1(t), ..., \delta_r(t))$.

To develop our results the following lemmas and property are used.

*Lemma 1.* (Congruence). Given matrix $P = P^T$ and a full column rank matrix $Q$, it holds that

$$P > 0 \quad \Rightarrow \quad QPQ^T > 0.$$

*Property 1.* (Schur complement) Let $\mathcal{M} = \mathcal{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, with $M_{11}$ and $M_{22}$ square matrices of appropriate dimensions. Then:

$$\begin{aligned} \mathcal{M} < 0 \Leftrightarrow & \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \\ \Leftrightarrow & \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases} \end{aligned} \quad (8)$$

## 3. MAIN RESULTS

Our goal is to develop sufficient conditions for observer design for system (5). The following observer is considered:

$$\begin{aligned} \dot{\hat{x}}(t) =& A_{zz_\tau}\hat{x}(t) + D_{zz_\tau}\hat{x}(t-\tau) \\ & + G_{zz_\tau}\psi(H\hat{x}(t)) + L_{zz_\tau}(y(t) - \hat{y}(t)), \\ \hat{y}(t) =& C\hat{x}(t) \end{aligned} \quad (9)$$

where $L_{zz_\tau}$ are the observer gains, $\hat{x}(t)$ the estimated states, and $\hat{y}(t)$ the estimated output. This observer form can be used if Assumption 1 holds, i.e., the premise variables $z$ and the delayed ones $z_\tau$ are known. Based on (5) and (9), the estimation error dynamic is:

$$\begin{aligned} \dot{e}(t) =& \dot{x}(t) - \dot{\hat{x}}(t) \\ =& A_{zz_\tau}\big(x(t) - \hat{x}(t)\big) + D_{zz_\tau}\big(x(t-\tau) - \hat{x}(t-\tau)\big) \\ & - L_{zz_\tau}\big(y(t) - \hat{y}(t)\big) + G_{zz_\tau}\big(\psi(Hx(t)) - \psi(H\hat{x}(t))\big) \\ =& (A_{zz_\tau} - L_{zz_\tau}C)e(t) + D_{zz_\tau}e(t-\tau) \\ & + G_{zz_\tau}\big(\psi(Hx(t)) - \psi(H\hat{x}(t))\big). \end{aligned} \quad (10)$$

Thanks to Assumption 2 we obtain:

$$\begin{aligned} \psi(Hx(t)) - \psi(H\hat{x}(t)) =& \delta(t)\,(Hx(t) - H\hat{x}(t)) \\ = \delta(t)H\,(x(t) - \hat{x}(t)) =& \delta(t)He(t), \end{aligned} \quad (11)$$

and for simplification we denote $\eta := He(t)$. This leads to the following form for (10):

$$\begin{aligned} \dot{e}(t) =& (A_{zz_\tau} - L_{zz_\tau}C)e(t) + D_{zz_\tau}e(t-\tau) \\ & + G_{zz_\tau}\delta(t)\eta \\ \eta =& He(t). \end{aligned} \quad (12)$$

To develop the observer design conditions, we consider the Lyapunov functional (Fridman, 2014):

$$\begin{aligned} V(t, e, \dot{e}) =& e^T(t)Pe(t) + \int_{t-h}^{t} e^T(s)Se(s)ds \\ & + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)R\dot{e}(s)dsd\theta \\ & + \int_{t-\tau}^{t} e^T(s)Qe(s)ds, \end{aligned} \quad (13)$$

where $P$, $S$, $R$, and $Q$ are symmetric, positive definite matrices.

The following result can be formulated:

*Theorem 1.* Consider the error dynamics (12), and assume that $\tau$ is differentiable, $\dot{\tau} \leq d$, $d \in [0, 1)$ is a given constant, $\tau \leq h$, $h > 0$ is the maximum time-delay. If there exist matrices $P = P^T > 0$, $R = R^T > 0$, $S = S^T > 0$, $S_{12}$, $Q = Q^T > 0$, $M = \text{diag}(m_1, ..., m_r) > 0$ and scalar $\epsilon > 0$, so that $\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \geq 0$ such that (14) holds, with

$$\begin{aligned} \Sigma_{33}^{14} =& -(1-d)Q - 2R + S_{12} + S_{12}^T \\ \nu(M) =& -2M\text{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r}) \end{aligned} \tag{15}$$

then (12) is asymptotically stable.

**Proof.** Consider (13), where $P = P^T > 0$, $R = R^T > 0$, $S = S^T > 0$, $Q = Q^T > 0$. The derivative of $V$ is

$$\begin{aligned} \dot{V}(t,e,\dot{e}) =& e^T(t)P\dot{e}(t) + \dot{e}^T(t)Pe(t) \\ &+ h^2\dot{e}^T(t)R\dot{e}(t) - h\int_{t-h}^{t} \dot{e}^T(s)R\dot{e}(s)ds \\ &+ e^T(t)[S+Q]e(t) - e^T(t-h)Se(t-h) \\ &- (1-\dot{\tau}(t))e^T(t-\tau)Qe(t-\tau), \end{aligned} \tag{16}$$

Using the reciprocally convex approach (Park et al., 2011), (Fridman, 2014)

$$-h\int_{t-h}^{t} \dot{e}^T(s)R\dot{e}(s)ds \leq -\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}^T \begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{17}$$

where $z_1 = e(t) - e(t-\tau), z_2 = e(t-\tau) - e(t-h)$ and $\begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \geq 0$, for some $S_{12} \in \mathbb{R}^{n_x \times n_x}$.

Using $\dot{\tau} \leq d$ and denoting

$$\chi := \begin{bmatrix} e(t) \\ e(t-h) \\ e(t-\tau) \\ \delta(t)\eta \end{bmatrix} \tag{18}$$

we obtain

$$\dot{V}(t,e,\dot{e}) \leq \chi^T\Delta\chi \tag{19}$$

where $\Delta$ is defined in (20) and $\mathcal{A} = A_{zz_\tau} - L_{zz_\tau}C$, $\Sigma_{33} = -(1-d)Q - 2R + S_{12} + S_{12}^T$.

Next we use Assumption 2 to determine relaxed conditions for $\dot{V} < 0$. Consider the inequality:

$$\chi^T\Delta\chi + \chi^T\theta\chi \leq 0, \tag{21}$$

where

$$\theta = \begin{bmatrix} \epsilon I & 0 & 0 & H^TM \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & \nu(M) \end{bmatrix} \tag{22}$$

and $M = \text{diag}(m_1, ..., m_r) > 0$.

Let us now examine $\chi^T\theta\chi$:

$$\begin{aligned} \chi^T\theta\chi =& \epsilon e(t)^Te(t) + 2e(t)^TH^TM\delta(t)\eta \\ &+ (\delta(t)\eta)^T\nu(M)\delta(t)\eta. \end{aligned} \tag{23}$$

Since $\eta = He(t)$, we have

$$\begin{aligned} -\chi^T\theta\chi =& -\epsilon e(t)^Te(t) - 2\eta^TM\delta(t)\eta - (\delta(t)\eta)^T\nu(M)\delta(t)\eta \\ =& -\epsilon\|e(t)\|^2 - 2\eta^T\left(M\delta(t) - \delta(t)^TM\text{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r})\delta(t)\right)\eta. \end{aligned} \tag{24}$$

Since $m_i > 0$ and $\left(1 - \delta_i(t)\frac{1}{b_i}\right) \geq 0$, the following holds:

$$2\eta^T\left(M\delta(t) - \delta(t)^TM\text{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r})\delta(t)\right)\eta \geq 0. \tag{25}$$

Finally, we obtain

$$-\chi^T\theta\chi \leq -\epsilon\|e(t)\|^2. \tag{26}$$

Therefore, $\chi^T\Delta\chi + \chi^T\theta\chi < 0$ involves $\dot{V} < 0$.

Consider the matrix inequality $\Delta + \theta < 0$ in (27). Applying the Schur complement on (27), we get (14).

**Remark 1**: Even if sum-relaxations are applied, the obtained conditions are BMIs due to the multiplication of decision variables $L^TP$ and $L^TR$.

Sufficient LMI conditions are formulated in the following corollary:

*Corollary 1.* Consider the error dynamics (12), with $\dot{\tau} \leq d$, $d \in [0, 1)$, $\tau \leq h$. If there exist matrices $P = P^T > 0$, $Q = Q^T > 0$, $S = S^T > 0$ $M = \text{diag}(m_1, ..., m_p) > 0$, $N_{ij}$, $i, j = 1, \ldots, s$ scalar $\epsilon > 0$, and

$$F_{ij} < 0, \tag{28}$$

where $F_{ij}$ is defined in (29), where

$$\begin{aligned} \Sigma_{11}^{29} =& PA_{ij} + A_{ij}^TP - N_{ij}C - C^TN_{ij}^T + S + Q - P + \epsilon I \\ \Sigma_{33}^{29} =& -(1-d)Q - 2P + S_{12} + S_{12}^T \\ \nu(M) =& -2M\text{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r}), \end{aligned} \tag{30}$$

then the error dynamics (12) is asymptotically stable. The observer gains can be recovered from $L_{ij} = P^{-1}N_{ij}$, $i, j = 1, ..., s$.

**Proof.** Consider (14). To obtain LMI conditions, let $R = P$ and denote $N_{zz_\tau} = PL_{zz_\tau}$. In this case, (14) can be written as (31), where $\Sigma_{11}^{31} = PA_{zz_\tau}^T + * - N_{zz_\tau}C - * + S + Q - P + \epsilon I$, $\Sigma_{33}^{31} = -(1-d)Q - 2P + S_{12} + S_{12}^T$.

## 4. EXAMPLE

To illustrate the conditions developed, first we discuss them on a numerical example and then compare them to another result (Lin et al., 2008) from the literature.

### 4.1 Numerical example

Consider the following nonlinear system:

$$\begin{bmatrix} (A_{zz_\tau} - L_{zz_\tau}C)^T P + * + S + Q - R + \epsilon I & S_{12} & PD_{zz_\tau} + R - S_{12} & PG_{zz_\tau} + H^T M & h(A_{zz_\tau} - L_{zz_\tau}C)^T R \\ * & -R - S & R - S_{12}^T & 0 & 0 \\ * & * & \Sigma_{33}^{14} & 0 & hD_{zz_\tau}^T R \\ * & * & * & \nu(M) & hG_{zz_\tau}^T R \\ * & * & * & * & -R \end{bmatrix} \leq 0 \quad (14)$$

$$\Delta = \begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + h^2 \mathcal{A}^T R \mathcal{A} + S + Q - R & S_{12} & (P + h^2 \mathcal{A}^T R) D_{zz_\tau} + R - S_{12} & (P + h^2 \mathcal{A}^T R) G_{zz_\tau} \\ * & -R - S & R - S_{12}^T & 0 \\ * & * & \Sigma_{33} + h^2 D_{zz_\tau}^T R D_{zz_\tau} & h^2 D_{zz_\tau}^T R G_{zz_\tau} \\ * & * & * & h^2 G_{zz_\tau}^T R G_{zz_\tau} \end{bmatrix} \quad (20)$$

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} + S + Q - R + \epsilon I & S_{12} & PD_{zz_\tau} + R - S_{12} & PB_{zz_\tau} G + H^T M \\ * & -R - S & R - S_{12}^T & 0 \\ * & * & \Sigma_{33} & -H^T M \\ * & * & * & \nu(M) \end{bmatrix} + \begin{bmatrix} h^2 \mathcal{A}^T R \mathcal{A} & 0 & h^2 \mathcal{A}^T R D_{zz_\tau} & h^2 \mathcal{A}^T R G_{zz_\tau} \\ * & 0 & 0 & 0 \\ * & * & h^2 D_{zz_\tau}^T R D_{zz_\tau} & h^2 D_{zz_\tau}^T R G_{zz_\tau} \\ * & * & * & h^2 G_{zz_\tau}^T R G_{zz_\tau} \end{bmatrix} \leq 0 \quad (27)$$

$$F_{ij} = \begin{bmatrix} PA_{ij} + A_{ij}^T P - N_{ij} C - C^T N_{ij}^T + S + Q - P + \epsilon I & S_{12} & PD_{ij} + P - S_{12} & PG_{ij} + H^T M & h(A_{ij}^T P - C^T N_{ij}^T) \\ * & -P - S & P - S_{12}^T & 0 & 0 \\ * & * & \Sigma_{33}^{29} & 0 & hD_{ij}^T P \\ * & * & * & \nu(M) & hG_{ij}^T P \\ * & * & * & * & -P \end{bmatrix} \quad (29)$$

$$\begin{bmatrix} \Sigma_{11}^{31} & S_{12} & PD_{zz_\tau} + P - S_{12} & PG_{zz_\tau} + H^T M & h(PA_{zz_\tau} - N_{zz_\tau} C)^T \\ * & -P - S & P - S_{12}^T & 0 & 0 \\ * & * & \Sigma_{33}^{31} & 0 & hD_{zz_\tau}^T P \\ * & * & * & \nu(M) & hG_{zz_\tau}^T P \\ * & * & * & * & -P \end{bmatrix} \leq 0 \quad (31)$$

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -3 & -0.5 \\ 0 & -3 - \cos(x_1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ \begin{bmatrix} 2 & 2 \\ 0.75 + 0.25\cos(x_1) & 3 + \cos(x_1(t-\tau)) \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ -0.675 - 0.125\cos(x_1) & -0.675 - 0.125\cos(x_1) \end{bmatrix} \\ &\quad (\alpha_1(x_1) + \alpha_2(x_2)), \\ y &= x_1 \end{aligned} \quad (32)$$

where $\alpha_1(x_1)$ and $\alpha_2(x_2)$ are two nonlinear functions which satisfy Assumption 2. For the simulations we consider

$$\alpha_1(v) = \alpha_2(v) = \cos^2(v) + v, \quad (33)$$

and the constants that satisfy Assumption 2 are $b_1 = b_2 = 2$, but the obtained results are valid for any other nonlinear functions which satisfy Assumption 2 with $b_1$ and $b_2$.

For the rest of the nonlinearities we use the sector nonlinearity approach (Ohtake et al., 2001) and obtain a TS model of form (5) with the local matrices:

$$A_{11} = A_{12} = \begin{bmatrix} -3 & -0.5 \\ 0 & -2 \end{bmatrix}, \quad A_{21} = A_{22} = \begin{bmatrix} -3 & -0.5 \\ 0 & -4 \end{bmatrix},$$
$$G_1 = \begin{bmatrix} 0 & 0 \\ -0.55 & -0.55 \end{bmatrix}, G_2 = \begin{bmatrix} 0 & 0 \\ -0.8 & -0.8 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (34)$$

$$C = [1 \ 0], \quad D_{11} = \begin{bmatrix} 2 & 2 \\ 0.5 & 2 \end{bmatrix}, D_{12} = \begin{bmatrix} 2 & 2 \\ 0.5 & 4 \end{bmatrix},$$
$$D_{21} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}, \quad (35)$$

and membership functions:

$$q_1(z) = \frac{1 - cos(z)}{2}, \ q_2(z) = 1 - q_1(z), \ z = x_1.$$

The delay we consider is $\tau(t) = 0.1 + 0.1\cos(t)$. Applying Corollary 1, the obtained matrices for $\tau(t) \leq h = 0.2$ and $\dot{\tau}(t) \leq d = 0.2$ are:

$$N_{11} = [2.17 \ 0.13], \quad N_{12} = [1.31 \ 0.62],$$
$$N_{21} = [2.52 \ 0.22], \quad N_{22} = [1.75 \ 0.51],$$
$$P = \begin{bmatrix} 0.65 & -0.25 \\ -0.25 & 0.16 \end{bmatrix}.$$

Next, we simulate the system. The initial points for the state vector and estimated state vector are $x_0 = [1 \ 2]^T$ and $\hat{x}_0 = [1 \ 0]^T$, respectively. It can be seen in Fig. 1 that the estimation error converges to zero. The individual states and their estimates are illustrated in Fig. 2 and Fig. 3.

Note that although for the simulations we used the nonlinearity (33), the conditions are satisfied for any known nonlinearity with bound $b \leq 2$.

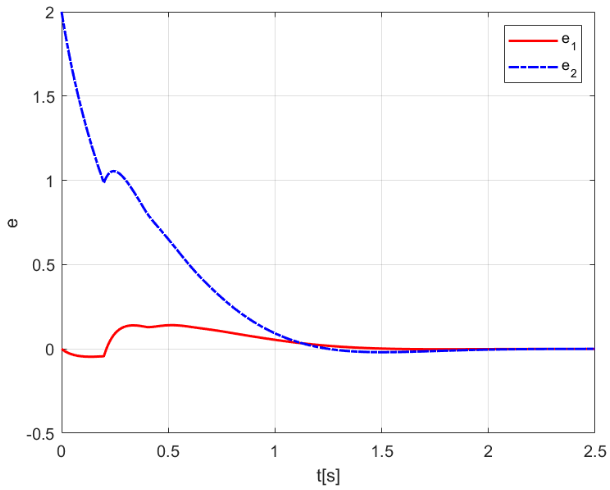


Fig. 1. Convergence of the estimation error

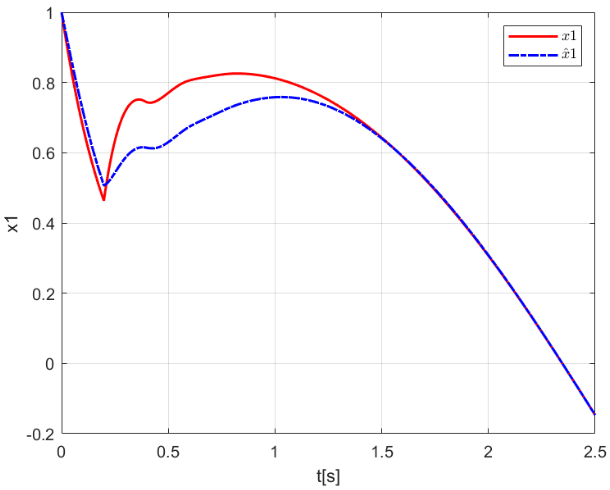


Fig. 2. True and estimated values of $x_1$

On the other hand, the larger the slope $b$ is, the conditions become harder to satisfy. The maximum slope for which our approach gives feasible solution is $b = 19$.

### *4.2 Comparison with (Lin et al., 2008)*

Next, we compare our approach to that of Lin et al. (2008), specifically Condition T1. This condition in (Lin et al., 2008) depends only on the derivative of the delay, while we use the bounds on both the delay and its derivative. On the other hand (Lin et al., 2008) considers a classic TS model, with linear consequents. For a fair comparison in designing the observer we use a reduced Lyapunov functional that takes into account only the bound on the delay. Consider the following nonlinear system:

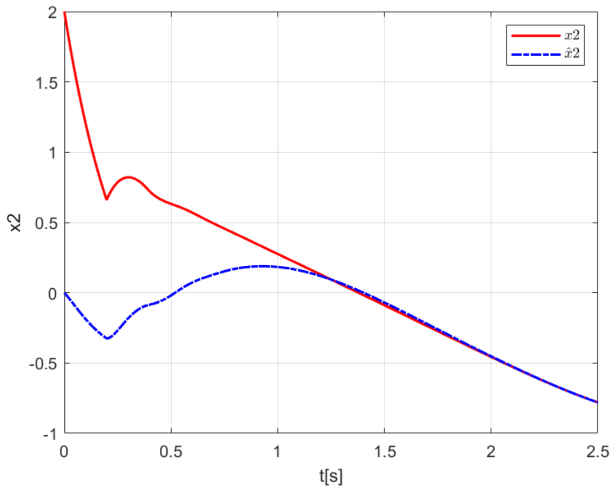


Fig. 3. True and estimated values of $x_2$

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-3 & -0.5\\ 0 & -2-\cos(x_1)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\\ &+\begin{bmatrix}2 & 2\\ 0.75+0.25\cos(x_1) & 3+\cos(x_1(t-\tau))\end{bmatrix}\begin{bmatrix}x_1(t-\tau)\\ x_2(t-\tau)\end{bmatrix}\\ &+\begin{bmatrix}0\\ -0.6\end{bmatrix}\alpha(x_2),\\ y &= x_1\end{aligned} \tag{36}$$

where $\alpha(x_2) = \cos^2(x_2) + x_2$. The constant that satisfies Assumption 2 is $b = 2$.

If a classic TS representation, i.e., without nonlinear consequents is considered, system (36) can be represented by an affine TS model, with one of the premise variables being $x_2$, that has to be estimated. One way to handle the model-observer mismatch is to assume that this mismatch is Lipschitz continuous, with $\mu$ being the Lipschitz constant. Another condition involving $\mu$ is also included next to the condition of Lin et al. (2008), similar to (Bergsten, 2001).

We test the approaches on the time-delayed TS model (5), with the local matrices (34)-(35) and

$$A_{11} = A_{12} = \begin{bmatrix}-3 & -0.5\\ 0 & -1.6\end{bmatrix}, \quad A_{21} = A_{22} = \begin{bmatrix}-3 & -0.5\\ 0 & -3.6\end{bmatrix},$$
$$H = [0\ 1], \quad G = \begin{bmatrix}0\\ -0.6\end{bmatrix}.$$

The approach in (Lin et al., 2008) is infeasible, while in our approach the maximum value for $d$, for which it gives feasible solution is $d = 0.359$.

## 5. CONCLUSIONS AND FUTURE WORK

In this paper we presented an observer design method to estimate the unknown states, using time-delay TS fuzzy models with nonlinear consequents. We assumed that the delay and its derivative are known and may be used in the observer. The design conditions were illustrated on a numerical example and in a comparison with another result from the literature.

It should be noted that the assumption that the delays are measured is extremely restrictive. In our future work we consider the case when the delay also has to be estimated.

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