Estimation for control – Practical Assignment 4
Continuous-time linear observer design

Logistics

- This practical assignment should be carried out by a group of maximum two students.
- The assignment solution consists of Matlab code and Simulink model. This code will be checked and run by the teacher during the lab class, and your attendance to the lab will only be registered if you have a working, original solution. Validated attendances for all the labs are necessary for eligibility to the exam. Moreover, at most two labs can be recovered at the end of the semester, which means accumulating three or more missing labs at any point during the semester automatically leads to final ineligibility.
- Discussing ideas among students is encouraged; however, directly sharing and borrowing pieces of code is forbidden, and any violation of this rule will lead to disqualification of the solution.

Assignment description

Prerequisite:

- A nonlinear model, with state vector $x = [x_1,\ x_2, ...x_n]^T$, and input vector $u = [u_1,\ u_2, ...u_m]^T$ where $n \geq 3$, is the number of state variables, $m \geq 1$, is the number of inputs, and the dynamic equation has the form:
$$\begin{aligned} \dot{x} =& f(x, u) \\ y =& Cx, \end{aligned} \tag{1}$$
where $f(x) = [f_1(x, u), f_2(x, u), ...f_n(x, u)]^T$ is a vector function with at least one nonlinear term.
- nonlinear model – implementation in Matlab-Simulink
- linear state-space model

The task in Laboratory 2 was to design a controller for the linear model, which then was applied also on the nonlinear model. The suggested type was a state-feedback control, and we assumed that all the states are available and we used them to design the control of the form, $u = -Kx$. In this assignment we will study the case when some of the states are not available and for them we will design a continuous-time linear observer. We will test the observer on both linear and nonlinear models.

Recap on observer design

We consider the case when we do not measure all the states, so the output matrix is not identity, $C \neq I$. As an example consider the following linear system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x, \end{aligned} \tag{2}$$

where $x = [x_1,\, x_2]^T$ is the state vector, u is the input, y is the output. The output of the system is $y = x_1$, which means we only measure x_1, but for a state-feedback control we need all the states, so we need also x_2. That is the reason why we are using an observer, which can be intuitively understood as an approximation of the model. We consider:

$$\begin{aligned} \hat{x} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{3}$$

where $\hat{x}$ is the estimate of x, and L is the observer gain. We denote the difference between the estimation and the true states with $e = x - \hat{x}$, and we want this estimation error to converge to 0. We can write the dynamics of this estimation error in the following form:

$$\begin{aligned} \dot{e} = \dot{x} - \dot{\hat{x}} &= Ax + Bu - (A\hat{x} + Bu + L(y - \hat{y})) \\ &= A(x - \hat{x}) + L(Cx - C\hat{x}) \\ &= (A - LC)e. \end{aligned} \tag{4}$$

If the pair (A, C) is observable, then the eigenvalues of $(A - LC)$ can be placed arbitrarily. In order to obtain stability, i.e. the estimation error, $e \to 0$, when $t \to \infty$, we need to place the eigenvalues on the left half plane. For the given example, we have:

$$A - LC = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -1 - l_2 & -0.4 \end{bmatrix}. \tag{5}$$

The characteristic polynomial of the observer is:

$$\det(sI - A + LC) = s^2 + (l_1 + 2)s + 2l_1 + l_2 + 1 = 0 \tag{6}$$

We need to find l_1, l_2 so that the roots $s_1, s_2 < 0$. We impose that $s_1 = -2$ and $s_2 = -3$, which leads to $l_1 = 3$, $l_2 = -1$, so $L = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. For this example it was easy to calculate l_1, l_2, but in general it might not be this simple. In Matlab the `place` function can be used: L^T `= place(`A^T`,`C^T`, [poles])`.

Requirements:

- Change your C matrix: $C \neq I$ (is not the identity matrix), $D = 0$ (D is still zero matrix)
- Check if linear system is fully state observable, by calculating the rank of the observability matrix:

$$P_o = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix}$$

- Find the observer gain L
- Test observer on both linear and nonlinear models, without control
- Use your state-feedback control with the estimated states
- Compare the estimated state, $\hat{x}$, with the true state, x

Hint: Useful Matlab functions `obsv`, `place`.