Contents

Chapter 1

Dynamic models - recapitulation

TODO: include values and simulations

1.1 System models

A **mathematical model** is an equation or set of equations that adequately describe the behavior of a system. A model is always an approximation of the real system. System models can be obtained in two ways:

- Identification: given sufficient data from experiments, a mathematical model can be found from the results $\Longrightarrow$ **System identification**.
- From first-principle: the model is built form physical/ chemical/ etc. laws $\Longrightarrow$ **System modeling**.

Some classification of models are:

- Linear: they are homogeneous and obey the principle of superposition, i.e., $\Sigma(au_1 + bu_2) = a\Sigma(u_1) + b\Sigma(u_2)$.
- Nonlinear: most system model, in particular those obtained by first principles modeling are nonlinear. For instance, in case of a hardening spring, the resonant frequency increases with the oscillation amplitude.

Example 1.1 *Consider the usual spring-mass damper system in Figure 1.1, which is generally used to model a shock absorber of a car. The model of the system is derived from Newton's second law, and the force equation is usually written as:*

$$M\frac{d^2y(t)}{dt^2} = -f\frac{dy}{dt} - ky(t) + \tau(t) \tag{1.1}$$

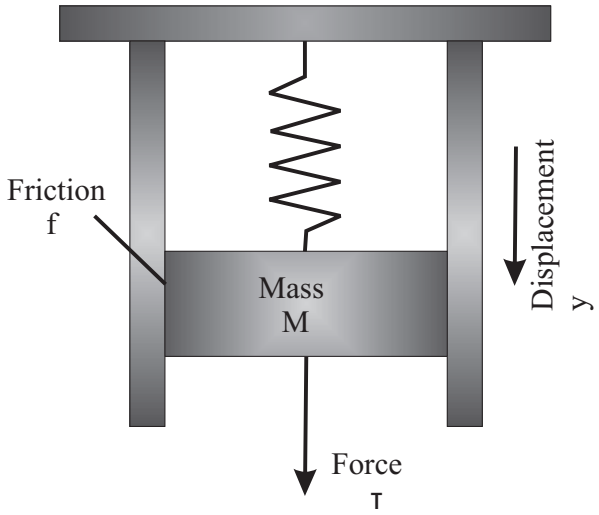

Figure 1.1: A spring-mass-damper system.

where M is the mass, f is the friction coefficient, k is the stiffness of the linear spring, $\frac{d^2y(t)}{dt^2}$ is the acceleration, $\frac{dy}{dt}$ is the velocity, $y(t)$ is the displacement, $\tau(t)$ is the input (force). Note that this system is linear.

If a so-called hardening spring is used, then the model is (see also Duffing equation):

$$M\frac{d^2y(t)}{dt^2} = -f\frac{dy}{dt} - k_l y(t) - k_n y^3(t) + \tau(t) \tag{1.2}$$

where the restoring force provided by the spring is given by $k_l y(t) + k_n y^3(t)$, which is nonlinear.

Models can be described by:

- Input-output models: a relationship between the input and output of the system. For linear, time-invariant systems classically transfer functions are used.
- State-space models: a relationship between the inputs, states (usually variables that have a physical meaning), and outputs (measurements). The description is usually in the form of system of differential (in continuous time) or difference (in discrete time) equations.

Recall that

Definition 1.1 *The transfer function of a linear system is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable, with **all the initial conditions assumed to be zero**.*

Example 1.2 *Consider the spring-mass damper system in Example 1.1.*

The output variable is y – since the displacement is measured – and the (control) input variable is $u := \tau$. Applying the Laplace transform in zero initial conditions gives

$$Ms^2Y(s) = -fsY(s) - kY(s) + U(s)$$

where s denotes the Laplace variable and $Y(s)$ and $U(s)$ are the Laplace transform of $y(t)$ and $\tau(t)$, respectively.

A transfer function description of the (1.1) *is obtained as*

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + fs + k}$$

Usually the state-space system model is expressed as a set of first order differential equations. Denoting $x_1 = y$, $x_2 = \frac{dy}{dt} = \dot{x}_1$, we have

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k}{M}x_1(t) - \frac{f}{M}x_2(t) + \frac{1}{M}u(t) \end{aligned} \tag{1.3}$$

where $\boldsymbol{x} = [x_1,\, x_2]^T$ is the vector of the state variables and u is the control input. Let us now see how the usual matrix form can be obtained. Model (1.3) *can be written as*

$$\begin{aligned} \dot{x}_1(t) &= [0,\, 1] && [x_1(t),\, x_2(t)]^T && + 0u(t) \\ \dot{x}_2(t) &= [-\frac{k}{M} - \frac{f}{M}] && [x_1(t),\, x_2(t)]^T && + \frac{1}{M}u(t) \end{aligned}$$

i.e., $\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$, with $\boldsymbol{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\dot{\boldsymbol{x}} = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f}{M} \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ \frac{1}{M} \end{pmatrix}$. For this system, the output – which is generally the measured variable(s) – is the displacement, $y(t) = x_1(t)$. Therefore, we have

$$y(t) = x_1(t) = [1,\, 0]\boldsymbol{x} = C\boldsymbol{x}$$

with $C = [0\,1]$.

The final description of the system is

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} \end{aligned}$$

which is the usual state-space model of the system. Note that for this particular case the throughput term – the D matrix – is zero, i.e., the input does not affect directly the output.

Remark: *Note that the state-space description* (1.1) *is not unique: choosing* $x_1 = y$, $x_2 = M\frac{dy}{dt}$ *we get*

$$\dot{x}_1(t) = \frac{1}{M}x_2(t)$$
$$\dot{x}_2(t) = -kx_1(t) - fx_2(t) + u(t)$$

which will lead to different matrices.

Remark: *transfer functions are classically used for linear, time-invariant systems. Although for nonlinear system sequences of transfer functions may be defined, the analysis is cumbersome and the results are scarce.*

The state-space model for (1.2) *can be obtained similar to* (1.1)*:*

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{k_l}{M}x_1(t) - \frac{k_n}{M}x_1^3(t) - \frac{f}{M}x_2(t) + \frac{1}{M}u(t)$$

Note however, that due to the nonlinear term $\frac{k_n}{M}x_1^3(t)$*, this system cannot be expressed in the usual matrix form with constant matrices.*

Deterministic nature:

- Deterministic: the output of the system can be determined at any time from the knowledge of the inputs up to that time. Both models (1.1) and (1.2) are deterministic.
- Stochastic: there is random behavior in the system, which will affect the output.

Variation with respect to time:

- Time invariant: the model does not depend on time. Models (1.1) and (1.2) are time invariant.
- Time-varying: if the mass M is considered to be the total mass of the car, including the fuel that is gradually burned, the models become time-varying (in fact parameter-varying).

Time-dependence:

- Continuous-time: most models developed from first principles are continuous time.
- Discrete-time: if the measurements are taken in discrete time and the control input is applied in discrete time (usual case in digital control), a discrete-time model of the system considered may be adequate.

1.2 Mathematical intermezzo

1.2.1 Linearization - Taylor series expansion

The easiest type of models to handle is the linear model. For such models, sufficient and necessary conditions for stability, stabilization, estimation, performance measures, etc. exist, together with a number of analysis and design conditions to establish the desired performances. Although most physical models are nonlinear, a linear approximation may be sufficient if the important variables (state, input, output) vary in a restricted domain. Such an approximation can be obtained using the Taylor series expansion.

Consider a continuous and differentiable function $f : \mathbb{R}^n \to \mathbb{R}$. The Taylor series expansion around an operating point $\boldsymbol{x}_0 = [x_{10},\, x_{20},\, \ldots,\, x_{n0}]^T$ is

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (\boldsymbol{x}-\boldsymbol{x}_0) + \frac{1}{2!}\,(\boldsymbol{x}-\boldsymbol{x}_0)^T \left.\frac{\partial^2 f}{\partial \boldsymbol{x}^2}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (\boldsymbol{x}-\boldsymbol{x}_0) + h.o.t.$$

Since we are interested in a linear approximation, the Taylor series above is truncated after the first term:

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0} (\boldsymbol{x}-\boldsymbol{x}_0) \tag{1.4}$$

From here, we have

$$\underbrace{f(\boldsymbol{x}) - f(\boldsymbol{x}_0)}_{\delta \boldsymbol{y}} \approx \underbrace{\left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_0}}_{A} \underbrace{(\boldsymbol{x}-\boldsymbol{x}_0)}_{\delta \boldsymbol{x}}$$

i.e.,

$$\delta \boldsymbol{y} = A \delta \boldsymbol{x}$$

which is a linear expression.

Example 1.3 *Consider the nonlinear state-space model for* (1.2)*, repeated here for convenience:*

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{k_l}{M} x_1(t) - \frac{k_n}{M} x_1^3(t) - \frac{f}{M} x_2(t) + \frac{1}{M} u(t)
\end{aligned}$$

The first equation is linear. The second one is nonlinear, but for a small enough $x_1(t)$ it can be approximated by a linear model. This model will be obtained by the Taylor series expansion. The operating point around which the linearization is performed is $x_1 = 0$, $x_2 = 0$, $u = 0$, $\dot{x}_2 = 0$.

Remark: *$x_1 = 0$, $x_2 = 0$, $u = 0$, and $\dot{x}_1$ satisfy the second equation, thus this point is indeed an operating point.*

The nonlinear function that will be linearized is the second equation of (1.2)*, i.e.,*

$$f_2(x_1, x_2, u, \dot{x}_2) = -\frac{k_l}{M}x_1 - \frac{k_n}{M}x_1^3 - \frac{f}{M}x_2 + \frac{1}{M}u - \dot{x}_2 = 0 \qquad (1.5)$$

Since for linearization the time-dependence of the variables is not of interest, to simplify the notation, the time-dependence in (1.5) *is omitted.*

Applying the Taylor series expansion, we get

$$f_2(x_1, x_2, u, \dot{x}_2) \approx f(0,0,0,0) + \left.\frac{\partial f}{\partial [x_1,\, x_2,\, u,\, \dot{x}_2]^T}\right|_{x_1=0,\, x_2=0,\, u=0, \dot{x}_2=0} \begin{pmatrix} x_1 - 0 \\ x_2 - 0 \\ u - 0 \\ \dot{x}_2 - 0 \end{pmatrix}$$

$$f_2(x_1, x_2, u, \dot{x}_2) \approx 0 + \left.\begin{pmatrix} -\frac{k_l}{M} - 3\frac{k_n}{M}x_1^2 & -\frac{f}{M} & \frac{1}{M} & -1 \end{pmatrix}\right|_{x_1=0,\, x_2=0,\, u=0\, \dot{x}_2=0} \begin{pmatrix} x_1 \\ x_2 \\ u \\ \dot{x}_2 \end{pmatrix}$$

$$f_2(x_1, x_2, u, \dot{x}_2) \approx \begin{pmatrix} -\frac{k_l}{M} & -\frac{f}{M} & \frac{1}{M} & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ u \\ \dot{x}_2 \end{pmatrix}$$

$$f_2(x_1, x_2, u, \dot{x}_2) \approx -\frac{k_l}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}u - \dot{x}_2$$

i.e.,

$$\dot{x}_2 = -\frac{k_l}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}u$$

Finally, the linearized model of (1.2) *can be expressed as*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{k_l}{M}x_1 - \frac{f}{M}x_2 + \frac{1}{M}u \end{aligned}$$

Note that linearization can be performed in any other point, but the state variables of the linearized model may differ from those of the original variables, which must be taken into account when returning – for analysis, estimation or control – to the original nonlinear system. To show this, let us linearize the model (1.2) *in the point $x_1 = 1$, $x_2 = 1$, $u = 2$. For this point, we have $\dot{x}_1 = 1$ and $\dot{x}_2 = -\frac{k_l}{M} - \frac{k_n}{M} - \frac{f}{M} + 2\frac{1}{M} = \dot{x}_{20}$. Applying the Taylor series expansion for*

$$f_2(x_1, x_2, u, \dot{x}_2) = -\frac{k_l}{M}x_1 - \frac{k_n}{M}x_1^3 - \frac{f}{M}x_2 + \frac{1}{M}u - \dot{x}_2 = 0$$

we get

$$f_2(x_1, x_2, u, \dot{x}_2) \approx f_2(1, 1, 2, \dot{x}_{20}) + \left.\frac{\partial f_2}{\partial [x_1,\ x_2,\ u,\ \dot{x}_2]^T}\right|_{x_1=1,\, x_2=1,\, u=2, \dot{x}_2=\dot{x}_{20}} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \\ \dot{x}_2 - \dot{x}_{20} \end{pmatrix}$$

$$f_2(x_1, x_2, u, \dot{x}_2) \approx 0 + \left.\begin{pmatrix} -\frac{k_l}{M} - 3\frac{k_n}{M} x_1^2 & -\frac{f}{M} & \frac{1}{M} & -1 \end{pmatrix}\right|_{x_1=1,\, x_2=1,\, u=2\, \dot{x}_2=\dot{x}_{20}} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \\ \dot{x}_2 - \dot{x}_{20} \end{pmatrix}$$

$$f_2(x_1, x_2, u, \dot{x}_2) \approx \begin{pmatrix} -\frac{k_l}{M} - 3\frac{k_n}{M} & -\frac{f}{M} & \frac{1}{M} & -1 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \\ \dot{x}_2 - \dot{x}_{20} \end{pmatrix}$$

Recall that $f_2(x_1, x_2, u, \dot{x}_2) = 0$*, thus we have*

$$\underbrace{\dot{x}_2 - \dot{x}_{20}}_{\delta \dot{x}_2} = \begin{pmatrix} -\frac{k_l}{M} - 3\frac{k_n}{M} & -\frac{f}{M} & \frac{1}{M} \end{pmatrix} \underbrace{\begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \end{pmatrix}}_{\delta a \boldsymbol{x}} \tag{1.6}$$

Note however that these variables are not consistent with those in the first equation, thus those also need to be changed. One can also "linearize" the first equation in a similar way in the corresponding operating points, i.e., $x_1 = 1$*,* $x_2 = 1$*,* $u = 2$*, and* $\dot{x}_1 = 1$*. The function is*

$$f_1(x_1, x_2, u, \dot{x}_1) = x_2 - \dot{x}_1 = 0$$

and we obtain

$$f_1(x_1, x_2, u, \dot{x}_1) \approx f_1(1, 1, 2, 1) + \left.\frac{\partial f_1}{\partial [x_1,\ x_2,\ u,\ \dot{x}_1]^T}\right|_{x_1=1,\, x_2=1,\, u=2, \dot{x}_1=1} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \\ \dot{x}_1 - 1 \end{pmatrix}$$

$$f_1(x_1, x_2, u, \dot{x}_1) \approx 0 + \left.\begin{pmatrix} 0 & 1 & 0 & -1 \end{pmatrix}\right|_{x_1=1,\, x_2=1,\, u=2\, \dot{x}_1=1} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \\ \dot{x}_1 - 1 \end{pmatrix}$$

Since $f_1(x_1, x_2, u, \dot{x}_1) = 0$*, we have*

$$\underbrace{\dot{x}_1 - 1}_{\delta \dot{x}_1} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \end{pmatrix}}_{\delta_{\boldsymbol{x}}} \tag{1.7}$$

Combining (1.7) *and* (1.6)*, one obtains:*

$$\dot{x}_1 - 1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \end{pmatrix}$$

$$\dot{x}_2 - \dot{x}_{20} = \begin{pmatrix} -\frac{k_l}{M} - 3\frac{k_n}{M} & -\frac{f}{M} & \frac{1}{M} \end{pmatrix} \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ u - 2 \end{pmatrix}$$

i.e.,

$$\delta\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{k_l}{M} - 3\frac{k_n}{M} & -\frac{f}{M} & \frac{1}{M} \end{pmatrix} \delta\boldsymbol{x}$$

1.2.2 Discretization - forward Euler difference

The simplest – although not always adequately precise – way to obtain an approximate discrete-time model when given a continuous-time one is using the forward Euler difference. This difference can also be derived based on the Taylor series expansion.

Consider at this point a function $f(t)$. The first order Taylor series expansion around a point t_0 is:

$$f(t) \approx f(t_0) + \left.\frac{df}{dt}\right|_{t=t_0} (t - t_0)$$

and assume that the time instants t and t_0 correspond to two consecutive sampling instants, i.e., $t_0 = kT_s$, $t = (k+1)T_s$, thus the difference $t - t_0 = T_s$ is the sampling period. Then,

$$f(t) - f(t_0) = \left.\frac{df}{dt}\right|_{t=t_0} (t - t_0)$$

i.e, the derivative $\left.\frac{df}{dt}\right|_{t=t_0}$ can be approximated as:

$$\left.\frac{df}{dt}\right|_{t=t_0} = \frac{f(t) - f(t_0)}{(t - t_0)} = \frac{f((k+1)T_s) - f(kT_s)}{T_s} \tag{1.8}$$

Remark: note that the geometrical interpretation of the Euler discretization is that the slope of the line that is tangent to a curve at a given point is equal to the derivative.

Remark: when choosing the sampling period T_s or the frequency ω_s, Shannon's sampling theorem should be kept in mind, which states that "A function $f(t)$ that has a bandwidth ω_b is uniquely determined by a discrete set of sample values provided that the sampling frequency is greater than $\omega_s = 2\omega_b$. The sampling frequency $\omega_s = 2\omega_b$ is the Nyquist frequency. In applications, a useful rule is to sample the signal at about 10 times higher than the highest frequency thought to be present.

Example 1.4 *Recall the model in Example 1.1, which is a continuous-time one. If a digital controller is used or an observer is needed to estimate the sates, it may be advantageous to design them based on a discrete-time model instead of a continuous one. Thus, the model needs to be discretized. In this example, we consider the linear model* (1.1)*. The nonlinear one* (1.2) *can be similarly discretized.*

For easier reference, recall that the model is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\frac{k}{M}x_1(t) - \frac{f}{M}x_2(t) + \frac{1}{M}u(t)\end{aligned}$$

Based on (1.8)*, the derivatives are approximated as:*

$$\begin{aligned}\dot{x}_1(t) &= \frac{x_1((k+1)T_s) - x_1(kT_s)}{T_s}\\ \dot{x}_2(t) &= \frac{x_2((k+1)T_s) - x_2(kT_s)}{T_s}\end{aligned}$$

leading to – in what follows, the argument kT_s will simply be denoted by k –:

$$\begin{aligned}\frac{x_1(k+1) - x_1(k)}{T_s} &= x_2(k)\\ \frac{x_2(k+1) - x_2(k)}{T_s} &= -\frac{k}{M}x_1(k) - \frac{f}{M}x_2(k) + \frac{1}{M}u(k)\end{aligned}$$

Keeping on "next" state on the left-hand side gives

$$\begin{aligned}x_1(k+1) &= T_s x_2(k) + x_1(k)\\ x_2(k+1) &= T_s(-\frac{k}{M}x_1(k) - \frac{f}{M}x_2(k)) + x_2(k) + T_s\frac{1}{M}u(k)\end{aligned}$$

i.e.,

$$\begin{aligned}x_1(k+1) &= \begin{pmatrix}1 & T_s\end{pmatrix}\boldsymbol{x}(k)\\ x_2(k+1) &= \begin{pmatrix}-T_s\frac{k}{M} & 1 - T_s\frac{f}{M}\end{pmatrix}\boldsymbol{x}(k) + T_s\frac{1}{M}u(k)\end{aligned}$$

or, in matrix form

$$\boldsymbol{x}(k+1) = A_d\boldsymbol{x}(k) + B_d\boldsymbol{u}(k)$$

with $A_d = \begin{pmatrix}1 & T_s\\ -T_s\frac{k}{M} & 1 - T_s\frac{f}{M}\end{pmatrix}$ *and* $B_d = \begin{pmatrix}0\\ T_s\frac{1}{M}\end{pmatrix}$.

Note that the system matrices are obtained as $A_d = I + T_sA$, $B_d = T_sB$, where A and B are the continuous-time state and input matrices, respectively. The output matrix C_d is the same as the continuous-time output matrix C, only the sampling of the measurements is realized in discrete-time, i.e.,

$$\boldsymbol{y}(k) = C_d\boldsymbol{x}(k)$$

1.3 State-space models

A standard methodology to formulate models of dynamical systems is by using linear vector spaces and first order differential equations. The essential idea behind this formulation is the concept of state, which represents the memory the dynamic system has of its past **?**. The state of a dynamic system is the minimum set of variables that fully describe the system and its response to any given set of inputs or initial conditions.

The state variables completely characterize the system's state at any time and based on these any set of output variables may be computed. Although the actual state variables are not unique, the number of states describing a given system is unique – barring model reduction techniques. State-space descriptions resulting from system modeling are usually formulated in terms of physical and measurable variables. One set of state variables may be transformed into another – with the price of using their physical meaning –, for instance in order to facilitate stability analysis or controller or observer design.

In the standard form the dynamical system is described by a set of first order differential (or difference equations in discrete-time), in which the time-derivative of each state is expressed as a function of other state variables, i.e.,

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t) \\
\dot{x}_2(t) &= f_2(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t) \\
&\vdots = \vdots \\
\dot{x}_{n_x}(t) &= f_{n_x}(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t)
\end{aligned} \tag{1.9}$$

where t is the time, x_i, $i = 1, 2, \ldots, n_x$ denote the state variables – usually chosen as variables with physical meaning – and u_i, $i = 1, 2, \ldots, n_u$ denote the input variables, including the control variables. A set of output variables is also defined as

$$\begin{aligned}
y_1(t) &= h_1(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t) \\
y_2(t) &= h_2(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t) \\
&\vdots = \vdots \\
y_{n_y}(t) &= h_{n_y}(x_1(t), x_2(t), \ldots, x_{n_x}(t), u_1(t), u_2(t), \ldots, u_{n_u}(t), t)
\end{aligned} \tag{1.10}$$

These are usually the measured variables.

Remark: In the sequel, whenever evident, the time-dependence of the variables will be omitted.

Equations (1.9) together with (1.10) are usually written shortly as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) \\
\boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u}, t)
\end{aligned} \tag{1.11}$$

which is a *state-space description* of the considered system. Note that a system described by a higher-order differential equation can always be written as a set of first-order differential equations, thus the form (1.11) can be considered standard.

In discrete time, the general expression of the dynamics is

$$\begin{aligned} \boldsymbol{x}(k+1) &= \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k), k) \\ \boldsymbol{y}(k) &= \boldsymbol{h}(\boldsymbol{x}(k), \boldsymbol{u}(k), k) \end{aligned} \tag{1.12}$$

where k denotes the current sample.

Remark: Although the next descriptions are given in continuous time, the same definitions are valid for discrete-time systems.

A dynamic system is *autonomous* if its dynamics only depends on the states, i.e.,

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x}) \end{aligned} \tag{1.13}$$

and non-autonomous otherwise.

A special case arises when the functions $\boldsymbol{f}$ and $\boldsymbol{h}$ are affine in $\boldsymbol{u}$ and (1.11) can be written in the matrix form

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(\boldsymbol{x}, t, \theta)\boldsymbol{x} + B(\boldsymbol{x}, t, \theta)\boldsymbol{u} \\ \boldsymbol{y} &= C(\boldsymbol{x}, t, \theta)\boldsymbol{x} + D(\boldsymbol{x}, t, \theta)\boldsymbol{u} \end{aligned} \tag{1.14}$$

where θ denotes the vector of parameters.

Depending on the exact dependencies, we can distinguish, among others, between *linear parameter varying* systems when the matrices depend on the parameters θ

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(\theta)\boldsymbol{x} + B(\theta)\boldsymbol{u} \\ \boldsymbol{y} &= C(\theta)\boldsymbol{x} + D(\theta)\boldsymbol{u} \end{aligned} \tag{1.15}$$

linear time varying systems when the matrices depend on the time t

$$\begin{aligned} \dot{\boldsymbol{x}} &= A(t)\boldsymbol{x} + B(t)\boldsymbol{u} \\ \boldsymbol{y} &= C(t)\boldsymbol{x} + D(t)\boldsymbol{u} \end{aligned} \tag{1.16}$$

linear time invariant (LTI) systems when the system matrices are constant:

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} + D\boldsymbol{u} \end{aligned} \tag{1.17}$$

This course in principle concerns observer design for LTI systems.