# Chapter 5

# Luenberger and reduced-order observers

**Remark:** Although the developments below are presented for continuous-time systems, the results also hold for discrete-time ones.

## 5.1 Luenberger observer

In what follows, we discuss observer design with a somewhat different description than in Chapter 4 and the idea that "almost any system is an observer".

Consider the unforced system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

and a driven system

$$\dot{\boldsymbol{z}} = F\boldsymbol{z} + H\boldsymbol{x}$$

Assume that there exists a transformation matrix $T$ such that $TA - FT = H$. Then, we have

$$\begin{aligned} \dot{\boldsymbol{z}} - T\dot{\boldsymbol{x}} &= F\boldsymbol{z} + H\boldsymbol{x} - TA\boldsymbol{x} \\ &= F\boldsymbol{z} - FT\boldsymbol{x} \\ &= F(\boldsymbol{z} - T\boldsymbol{x}) \end{aligned}$$

The solution of this equation is

$$\boldsymbol{z} - T\boldsymbol{x} = e^{Ft}(\boldsymbol{z}(0) - T\boldsymbol{x}(0)) \tag{5.1}$$

or

$$\boldsymbol{z} = T\boldsymbol{x} + e^{Ft}(\boldsymbol{z}(0) - T\boldsymbol{x}(0))$$

Thus, if $\boldsymbol{z}(0) = T\boldsymbol{x}(0)$, then $\boldsymbol{z} = T\boldsymbol{x}$, $\forall t \geq 0$. Furthermore, if $\boldsymbol{z}(0) \neq T\boldsymbol{x}(0)$, but all the eigenvalues of $F$ have negative real parts, then $\boldsymbol{z} \to \boldsymbol{x}$ as $t \to \infty$.

**Remark:** The linear matrix equality $TA - FT = H$ has a unique solution if the matrices $A$ and $F$ have no common eigenvalues.

**Remark:** In case of input, i.e.,

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

it can also be included in the driven system:

$$\dot{\boldsymbol{z}} = F\boldsymbol{z} + H\boldsymbol{x} + TB\boldsymbol{u}$$

leading to the same equations:

$$\begin{aligned}
\dot{\boldsymbol{z}} - T\dot{\boldsymbol{x}} &= F\boldsymbol{z} + H\boldsymbol{x} + TB\boldsymbol{u} - TA\boldsymbol{x} - TB\boldsymbol{u} \\
&= F\boldsymbol{z} - FT\boldsymbol{x} \\
&= F(\boldsymbol{z} - T\boldsymbol{x})
\end{aligned}$$

Let us now consider the general case:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\
\boldsymbol{y} &= C\boldsymbol{x}
\end{aligned} \tag{5.2}$$

and the driven system

$$\dot{\boldsymbol{z}} = F\boldsymbol{z} + G\boldsymbol{y} + TB\boldsymbol{u}$$

Then we have

$$\begin{aligned}
\dot{\boldsymbol{z}} - T\dot{\boldsymbol{x}} &= F\boldsymbol{z} + GC\boldsymbol{x} + TB\boldsymbol{u} - TA\boldsymbol{x} - TB\boldsymbol{u} \\
&= F\boldsymbol{z} + (GC - TA)\boldsymbol{x}
\end{aligned}$$

If $T$ is such that $GC - TA = -FT$, then

$$\begin{aligned}
\dot{\boldsymbol{z}} - T\dot{\boldsymbol{x}} &= F\boldsymbol{z} + (GC - TA)\boldsymbol{x} \\
&= F\boldsymbol{z} - FT\boldsymbol{x} \\
&= F(\boldsymbol{z} - T\boldsymbol{x})
\end{aligned}$$

which has the solution (5.1).

Note that up until this point, it is not even required that the dimensions of the observed system (5.2) and that of the driven system (5.1) is the same. This fact can be exploited to design reduced order observers for the case when only some of the states need to be estimated.

A convenient observer is the *identity observer*, i.e., when the objective is to estimate the whole state vector $\boldsymbol{x}$ and $T = I$. In such a case, consider the dynamic system (5.2) and the observer (5.1). Since $TA - FT = GC$, and the matrix $T$ is fixed, $T = I$, we have $F = A - GC$ and the dynamics of $\boldsymbol{z}$ are given by

$$\dot{\boldsymbol{z}} = (A - GC)\boldsymbol{z} + G\boldsymbol{y}$$

i.e., it will only depend on the matrix $G$.

Naturally, all the above are possible if and only if the observability matrix $\Gamma_o$ has full column rank.

**Example 5.1** *Consider a dynamic system of the form* (5.2)*, with the matrices*

$$A = \begin{pmatrix} -1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

***todo***

**todo: compare with the observer in the previous chapter**

## 5.2 Reduced order observers

The previous idea can also be used to design reduced-order observers, i.e., observers that do not reconstruct the whole state vector, but only a part of it. Such a reconstruction can be easily motivated when some states are measured, i.e., the output matrix is of the form $C = \begin{pmatrix} I & 0 \end{pmatrix}$. It is then convenient to reconstruct those states that are not measured.

**Remark:** If $C$ is not of the form $C = \begin{pmatrix} I & 0 \end{pmatrix}$, an appropriate change of coordinated $M = \begin{pmatrix} C \\ D \end{pmatrix}$, with the matrix $D$ selected such that $M$ nonsingular and $\bar{\boldsymbol{x}} = M\boldsymbol{x}$ can be used to transform it.

Consider then the system

$$\begin{aligned} \dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{x} = \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{pmatrix} = \boldsymbol{x}_1 \end{aligned}$$

which is assumed to be observable, i.e., the observability matrix $\Gamma_o$ has full rank.

The system can be partitioned as:

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 + B_1\boldsymbol{u} \\ \dot{\boldsymbol{x}}_2 &= A_{21}\boldsymbol{x}_1 + A_{22}\boldsymbol{x}_2 + B_2\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{x}_1 \end{aligned}$$

leading to

$$\begin{aligned} \dot{\boldsymbol{y}} &= A_{11}\boldsymbol{y} + A_{12}\boldsymbol{x}_2 + B_1\boldsymbol{u} \\ \dot{\boldsymbol{x}}_2 &= A_{21}\boldsymbol{y} + A_{22}\boldsymbol{x}_2 + B_2\boldsymbol{u} \end{aligned} \tag{5.3}$$

Note that if $(A, C)$ is an observable pair, then $(A_{22}, A_{12})$ is also observable.

The main idea is that since $\boldsymbol{y}$ is known, then $\dot{v}y$ is known, thus $A_{12}\boldsymbol{x}_2$ is known, and $\boldsymbol{x}_2$ could be estimated from the second equation of (5.3). In fact, one can rewrite (5.3) as

$$\begin{aligned} A_{12}\boldsymbol{x}_2 &= \dot{\boldsymbol{y}} - A_{11}\boldsymbol{y} - B_1\boldsymbol{u} \\ \dot{\boldsymbol{x}}_2 &= A_{21}\boldsymbol{y} + A_{22}\boldsymbol{x}_2 + B_2\boldsymbol{u} \end{aligned} \tag{5.4}$$

Define the observer as

$$\dot{\widehat{\boldsymbol{x}}}_2 = A_{22}\widehat{\boldsymbol{x}}_2 + B_2\boldsymbol{u} + A_{21}\boldsymbol{y} + L(\dot{\boldsymbol{y}} - A_{11}\boldsymbol{y} - B_1\boldsymbol{u} - A_{12}\widehat{\boldsymbol{x}}_2) \tag{5.5}$$

with $L$ the observer gain to be designed and the estimation error $\boldsymbol{e}_2 = \boldsymbol{x}_2 - \widehat{\boldsymbol{x}}_2$. The estimation error dynamics are given by

$$\begin{aligned} \dot{\boldsymbol{e}}_2 &= A_{21}\boldsymbol{y} + A_{22}\boldsymbol{x}_2 + B_2\boldsymbol{u} - \\ &\quad - \left( A_{22}\widehat{\boldsymbol{x}}_2 + B_2\boldsymbol{u} + A_{21}\boldsymbol{y} + L(\underbrace{\dot{\boldsymbol{y}} - A_{11}\boldsymbol{y} - B_1\boldsymbol{u}}_{A_{12}\boldsymbol{x}_2} - A_{12}\widehat{\boldsymbol{x}}_2) \right) \\ &= (A_{22} - LA_{12})\boldsymbol{e}_2 \end{aligned}$$

Thus, if $L$ is chosen such that $(A_{22} - LA_{12})$ is Hurwitz, then the estimation error dynamics are asymptotically stable.

Note however, that this observer requires the derivative of the output, $\dot{\boldsymbol{y}}$, which may be problematic to obtain. A way to avoid the derivative is presented in what follows.

Let us reorder the terms in (5.5) and rewrite $\dot{\widehat{\boldsymbol{x}}}_2$ as

$$\dot{\widehat{\boldsymbol{x}}} = (A_{22} - LA_{12})\widehat{\boldsymbol{x}}_2 + (B_2 - LB_1)\boldsymbol{u} + (A_{21} - LA_{11})\boldsymbol{y} + L\dot{\boldsymbol{y}}$$

and define $\boldsymbol{z} = \widehat{\boldsymbol{x}}_2 - L\boldsymbol{y}$. Then,

$$\begin{aligned} \dot{\boldsymbol{z}} &= \dot{\widehat{\boldsymbol{x}}}_2 - L\dot{\boldsymbol{y}} \\ &= (A_{22} - LA_{12})\underbrace{\widehat{\boldsymbol{x}}_2}_{\boldsymbol{z}+L\boldsymbol{y}} + (B_2 - LB_1)\boldsymbol{u} + (A_{21} - LA_{11})\boldsymbol{y} + L\dot{\boldsymbol{y}} - L\dot{\boldsymbol{y}} \\ &= (A_{22} - LA_{12})\boldsymbol{z} + (B_2 - LB_1)\boldsymbol{u} + (A_{21} - LA_{11})\boldsymbol{y} + (A_{22} - LA_{12})L\boldsymbol{y} \end{aligned}$$

i.e., it does not depend on $\dot{v}y$.

Exploiting this fact, the observer can be written as

$$\begin{aligned}\dot{\boldsymbol{z}} &= (A_{22} - LA_{12})\boldsymbol{z} + (B_2 - LB_1)\boldsymbol{u} + (A_{21} - LA_{11})\boldsymbol{y} + (A_{22} - LA_{12})L\boldsymbol{y} \\ \widehat{\boldsymbol{x}}_2 &= \boldsymbol{z} + L\boldsymbol{y}\end{aligned} \tag{5.6}$$

and therefore the differentiation of $\boldsymbol{y}$ is avoided.

**Example 5.2** *Consider the dynamic system in Example 5.1. Since the first state is measured, we design an observer that estimates the remaining ones. The system can be partitioned as*

$$A = \left(\begin{array}{c|cc} -1 & 1 & 2 \\ \hline 2 & 1 & 3 \\ 1 & 3 & -4 \end{array}\right) = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right)$$

$$B = \left(\begin{array}{c} 0 \\ \hline 0 \\ 1 \end{array}\right) \qquad C = \left(\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right)$$

*The system is observable, and the* $(A_{22}, A_{12})$ *is also observable,* $\Gamma_o(A_{22}, A_{12}) = \begin{pmatrix} 1 & 2 \\ 7 & -5 \end{pmatrix}$, $\operatorname{rank}(\Gamma_o(A_{22}, A_{12})) = 2$*. Let us choose* $L = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$*. With this observer gain, the error dynamics are*

$$\dot{\widehat{\boldsymbol{x}}} = \begin{pmatrix} -1 & -1 \\ 0 & -10 \end{pmatrix}$$

*having the poles* $-1$*,* $-10$*, i.e., it is asymptotically stable. The observer is given by*

$$\begin{aligned}\dot{\boldsymbol{z}} &= \begin{pmatrix} -1 & -1 \\ 0 & -10 \end{pmatrix}\boldsymbol{z} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\boldsymbol{u} + \begin{pmatrix} -1 \\ -26 \end{pmatrix}\boldsymbol{y} \\ \widehat{\boldsymbol{x}}_2 &= \boldsymbol{z} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}\boldsymbol{y}\end{aligned}$$

***todo: sim***