Estimation for control – Practical Assignment 9
Unknown input decoupling

Logistics

- This practical assignment should be carried out by a group of maximum two students.
- The assignment solution consists of Matlab code and Simulink model. This code will be checked and run by the teacher during the lab class, and your attendance to the lab will only be registered if you have a working, original solution. Validated attendances for all the labs are necessary for eligibility to the exam. Moreover, at most two labs can be recovered at the end of the semester, which means accumulating three or more missing labs at any point during the semester automatically leads to final ineligibility.
- Discussing ideas among students is encouraged; however, directly sharing and borrowing pieces of code is forbidden, and any violation of this rule will lead to disqualification of the solution.

Unknown input decoupling observer design

In this laboratory we consider the case when an unknown – not necessarily constant – disturbance affects the system. Consider the continuous-time systems:

$$\begin{aligned}\dot{x} =& A\,x + B\,u + E\,d \\ y =& C\,x,\end{aligned} \tag{1}$$

where $d \in \mathbb{R}^{n_d}$ is the disturbance and E is the disturbance distribution matrix. The (Luenberger) observer has the form:

$$\begin{aligned}\dot{z} =& F\,z + TB\,u + K\,y \\ \hat{x} =& z + H\,y,\end{aligned} \tag{2}$$

where z is an auxiliary state, $\hat{x}$ is the estimate of x and F, T, K and H are matrices of appropriate dimensions to be designed. The objective here is to estimate the system states, which means that $e = x - \hat{x} \to 0$, when $t \to \infty$. The error dynamics can be written as:

$$\begin{aligned}\dot{e} =& \dot{x} - \dot{\hat{x}} = \ Ax + Bu + Ed - \left[\dot{z} + H\dot{y}\right] = \\ & Ax + Bu + Ed - \left[Fz + TBu + Ky + HC\dot{x}\right] = \\ & Ax + Bu + Ed - \left[Fz + TBu + Ky + HC(Ax + Bu + Ed)\right]\end{aligned} \tag{3}$$

We separate the term $K = K_1 + K_2$, so that we have $Ky = K_1Cx + K_2\,y$. Rearranging the terms we obtain:

$$\dot{e} = (A - K_1C - HCA)x + (B - TB - HCB)u + (E - HCE)d - Fz - K_2\,y. \tag{4}$$

Next, we compute the error dynamics:

$$\begin{aligned}\dot{e} =& (A - K_1C - HCA)x {\color{red}- (A - K_1C - HCA)\hat{x} + (A - K_1C - HCA)\hat{x}} \\ & + (B - TB - HCB)u + (E - HCE)d - Fz - K_2\,y = \\ & (A - K_1C - HCA)e + (A - K_1C - HCA)\underbrace{(z + Hy)}_{\hat{x}} \\ & + (B - TB - HCB)u + (E - HCE)d - Fz - K_2\,y = \\ & (A - K_1C - HCA)e + (A - K_1C - HCA - F)z + \big((A - K_1C - HCA)H - K_2\big)\,y \\ & + (B - TB - HCB)u + (E - HCE)d.\end{aligned} \tag{5}$$

which should be autonomous, i.e.,

$$\dot{e} = (A - K_1C - HCA)e, \tag{6}$$

This can be obtained by imposing

$$\begin{aligned} A - K_1C - HCA - F =& 0 \\ (A - K_1C - HCA)H - K_2 =& 0 \\ B - TB - HCB =& 0 \\ E - HCE =& 0, \end{aligned} \tag{7}$$

resulting in

$$\begin{aligned} F =& TA - K_1C \\ K_2 =& FH \\ T =& I - HC \\ H =& E(CE)^*, \end{aligned} \tag{8}$$

where $(CE)^*$ is the pseudo-inverse of CE, which means $CE(CE)^* = I$ (Matlab `pinv(CE)`). Note that if conditions (8) are satisfied, the estimation error e converges to 0 as $t \to \infty$ independent of the unknown input.

To satisfy (8), two conditions have to be verified:

- $\text{rank}(E) = \text{rank}(CE)$
- the pair (TA, C) should be fully observable.

Example

Consider the following linear model:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{\text{A}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{B}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0.5 \end{bmatrix}}_{\text{E}} d \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{C}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned} \tag{9}$$

The disturbance affects both x_1 and x_3. The rank condition is fulfilled $\text{rank}(E) = \text{rank}(CE) = 1$, so based on (8) we obtain:

$$H = \begin{bmatrix} 0.94 & 0.23 \\ 0 & 0 \\ 0.23 & 0.05 \end{bmatrix}, \quad T = \begin{bmatrix} 0.05 & 0 & -0.23 \\ 0 & 1 & 0 \\ -0.23 & 0 & 0.94 \end{bmatrix}. \tag{10}$$

The (TA, C) pair is observable, which leads to:

$$K_1 = \begin{bmatrix} 1.97 & 0.32 \\ -2.04 & -0.02 \\ -0.16 & -0.86 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -1.82 & -0.45 \\ 0.04 & 0.01 \\ -0.08 & -0.02 \end{bmatrix}, \quad F = \begin{bmatrix} -1.97 & 0.05 & 0.14 \\ 0.04 & -3 & 0.02 \\ 0.16 & -0.23 & -1.02 \end{bmatrix}. \tag{11}$$

The obtained unknown input decoupling observer is:

$$\dot{z} = \underbrace{\begin{bmatrix} -1.97 & 0.05 & 0.14 \\ 0.04 & -3 & 0.02 \\ 0.16 & -0.23 & -1.02 \end{bmatrix}}_{F} z + \underbrace{\begin{bmatrix} 0.05 & 0 & -0.23 \\ 0 & 1 & 0 \\ -0.23 & 0 & 0.94 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 0 & 0 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}}_{B} u$$

$$+ \left(\underbrace{\begin{bmatrix} 1.97 & 0.32 \\ -2.04 & -0.02 \\ -0.16 & -0.86 \end{bmatrix}}_{K_1} + \underbrace{\begin{bmatrix} -1.82 & -0.45 \\ 0.04 & 0.01 \\ -0.08 & -0.02 \end{bmatrix}}_{K_2} \right) y \tag{12}$$

$$\hat{x} = z + \underbrace{\begin{bmatrix} 0.94 & 0.23 \\ 0 & 0 \\ 0.23 & 0.05 \end{bmatrix}}_{H} y,$$

Requirements:

- Design a linear continuous-time unknown input decoupling observer
- Test the obtained observer with a sinusoidal disturbance input d
- Plot the error (e) and the residual (r)
- Test the obtained observer with a piece-wise constant disturbance d

Hint: Useful Matlab functions `obsv`, `place`.