

# Estimation for control

## Laboratory guide

### 1 Introduction

This laboratory guide is developed to give a better understanding on linear and non-linear state estimation problems. All the theoretical concepts are presented through a case study, with the main core, a water tank example. In some cases, to highlight certain aspects other examples are presented. We start by presenting the nonlinear model of the tank system, which is followed by a linearization around an equilibrium point. First, a linear state-feedback control is designed for stabilization, under the assumption that all states are measured. Afterwards, an observer is designed to estimate the unmeasured states and the controller will use the estimated states.

To develop our results we will use the following notations:  $w_0$  means the equilibrium point of vector  $w \in \mathbb{R}^n$  around which we will do the linearization;  $w_{k0} \in \mathbb{R}$  is the  $k$ -th element in vector  $w$  and 0 refers to the equilibrium point.  $I$  and  $0$  mean the identity and zero matrices of appropriate dimensions.  $\text{eig}(F)$  means the eigenvalues of  $F$ , and  $\text{rank}(F)$  refers to the rank of  $F$ . The notation  $f(x, u)$  refers to a vector function of the form  $f(x, u) = [f_1(x, u), f_2(x, u), \dots, f_n(x, u)]^T$ , where  $n$  is the number of states. For the sake of simplicity we will provide every numerical value with two decimal precision, and the time dependency of the states is not written, i.e.  $w(t)$  is denoted with  $w$ .

### Tank system

Consider the two tank system shown in Figure 1. Water is pumped from a reservoir into the upper tank. From this tank, the water flows to the lower tanks and then back to the reservoir. There is another pipe which pumps directly the water from the reservoir to the lower tank. The system has two control inputs  $u_1$  and  $u_2$ , which are the voltages applied to the motors of the pumps. It is assumed that the levels in the tanks ( $h_1$  and  $h_2$ ) are measured. The goal is to control the levels in the tanks.

### 2 Tank model

The equations describing the dynamics are:

$$\begin{aligned}\delta_1 \dot{F}_{in,1} &= -F_{in,1} + Q_{s,1} \cdot u_1 \\ \dot{h}_1 &= \frac{F_{in,1}}{A_1} - \frac{s_1 \sqrt{2gh_1}}{A_1} \\ \delta_2 \dot{F}_{in,2} &= -F_{in,2} + Q_{s,2} \cdot u_2 \\ \dot{h}_2 &= \frac{s_1 \sqrt{2gh_1}}{A_2} - \frac{s_2 \sqrt{2gh_2}}{A_2} + \frac{F_{in,2}}{A_2}\end{aligned}\tag{1}$$

The parameter values are presented in Table 1. We denote the state vector with  $x = [x_1 \ x_2 \ x_3 \ x_4]^T$ , where  $x_1 = F_{in,1}$ ,  $x_2 = h_1$ ,  $x_3 = F_{in,2}$  and  $x_4 = h_2$ ; and the input vector is  $u = [u_1 \ u_2]^T$ . With these

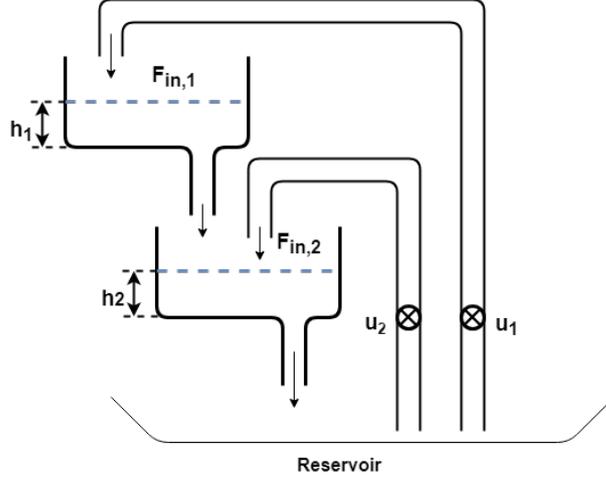


Figure 1: Cascaded tanks system.

Table 1: Parameter values used.

Parameter	Symbol	Value	Units
Acceleration	$g$	981	$\text{cm/s}^2$
Area of the tanks	$A_1, A_2,$	25, 16	$\text{cm}^2$
Outlet area of tank 1	$s_1$	1	$\text{cm}^2$
Outlet area of tank 2	$s_2$	1.5	$\text{cm}^2$
Input to flow gains	$Q_{s,1}, Q_{s,2}$	5, 10	$\text{cm}^3/\text{s}/\text{V}$
Motor time constants	$\delta_1, \delta_2$	0.5, 1	s

notations the nonlinear state-space model is:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\delta_1}x_1 + \frac{Q_{s,1}}{\delta_1}u_1 \\ \frac{x_1}{A_1} - \frac{s_1\sqrt{2gx_2}}{A_1} \\ \frac{1}{\delta_2}x_3 + \frac{Q_{s,2}}{\delta_2}u_2 \\ \frac{s_1\sqrt{2gx_2}}{A_2} - \frac{s_2\sqrt{2gx_4}}{A_2} + \frac{x_3}{A_3} \end{bmatrix} = \begin{bmatrix} f_1(x_1, u_1) \\ f_2(x_1, x_2) \\ f_3(x_3, u_2) \\ f_4(x_2, x_3, x_4) \end{bmatrix} = f(x, u) \quad (2)$$

The dynamics described in (2) is nonlinear. To design a linear controller, we need to linearize the system around an equilibrium point. In this context the linearization around 0, means that the level in the tanks are zero, which is not a normal operating point. Instead, we consider the scenario where the levels in the tanks vary around  $x_{20} = h_{10} = 5 [\text{cm}]$  and  $x_{40} = h_{20} = 3 [\text{cm}]$ .

In order to maintain these levels in the tanks we need to provide constant water inputs, otherwise, if the inputs are 0 for the tanks, the water will flow out, and eventually after a while the tanks will be empty. So we need to find the constant inputs for which the desired levels are maintained.

Based on [1] an equilibrium point for a nonlinear dynamic system of the form  $\dot{x} = f(x, u)$  can be found

in the point where the dynamic system of equations is 0, i.e.  $f(x_0, u_0) = 0$ . For (2) we have:

$$-x_1 + Q_{s,1}u_1 = 0 \quad (3)$$

$$\frac{x_1}{A_1} - \frac{s_1\sqrt{2gx_2}}{A_1} = 0 \quad (4)$$

$$-x_3 + Q_{s,2}u_2 = 0 \quad (5)$$

$$\frac{s_1\sqrt{2gx_2}}{A_2} - \frac{s_2\sqrt{2gx_4}}{A_2} + \frac{x_3}{A_2} = 0. \quad (6)$$

Based on (4),  $x_{10} = s_1\sqrt{2gx_{20}}$ , from where using (3) we obtain  $u_{10} = \frac{x_{10}}{Q_{s,1}}$ . Similarly, based on (6) we have  $x_{30} = s_2\sqrt{2gx_{40}} - s_1\sqrt{2gx_{20}}$ , and using (5) we obtain  $u_{20} = \frac{x_{30}}{Q_{s,2}}$ . Using the parameter values from Table 1 and the desired levels,  $x_{20} = 5 [cm]$  and  $x_{40} = 3 [cm]$ , we obtain:  $x_{10} = 99.04$ ,  $u_{10} = 19.8$ ,  $x_{30} = 16.03$  and  $u_{20} = 1.6$ .

Based on these notation the equilibrium point around which we do the linearization is:

$$u_0 = [19.8 \ 1.6]^T, \quad x_0 = [99.05 \ 5 \ 16.03 \ 3]^T \quad (7)$$

These values need to be considered, when we will compare the non-linear model with the linear model.

### 2.0.1 Possible difficulties with dynamic models

- A possible difficulty what you might encounter during your analysis is when the dynamic equation is not written in the classical form, like in (2). First problem is when the dynamic equation is given with the second derivative, let us consider the following example:

$$\ddot{\alpha} = \sin(\alpha) - \dot{\alpha} + u. \quad (8)$$

Based on [1], any  $n$  order differential equation can be transformed into a system of equations with  $n$  first order differential equations. In the case of (8) we can denote the states in the following way:  $x_1 = \alpha$ ,  $x_2 = \dot{\alpha}$ ; which leads to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_1) - x_2 + u. \end{aligned} \quad (9)$$

- Another possible difficulty can happen, when the dynamic equation is expressed in the following way:

$$\begin{aligned} 2\ddot{\theta} + \cos(\theta)\ddot{\psi} &= \sin(\theta) + \dot{\theta} + u_1 \\ \ddot{\theta} + 4\ddot{\psi} &= \cos(\psi) + \dot{\psi} + u_2. \end{aligned} \quad (10)$$

In this case we cannot directly express just  $\ddot{\theta}$  or  $\ddot{\psi}$ , because they depend on each other, but we can rewrite the equation in a matrix form:

$$\begin{bmatrix} 2 & \cos(\theta) \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \sin(\theta) + \dot{\theta} + u_1 \\ \cos(\psi) + \dot{\psi} + u_2 \end{bmatrix} \quad (11)$$

The matrix:  $\begin{bmatrix} 2 & \cos(\theta) \\ 1 & 4 \end{bmatrix}$  is non-singular, so it is invertible, and by multiplying from the left with the inverse the following is obtained:

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 2 & \cos(\theta) \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} \sin(\theta) + \dot{\theta} + u_1 \\ \cos(\psi) + \dot{\psi} + u_2 \end{bmatrix}. \quad (12)$$

From this point, (12) is in the same form as (8), so it can be easily transformed into the classical form, by transforming the second order differential terms into first order terms like in the first bullet point.

## 2.1 Linearized model

Recall that the truncated Taylor series approximation for a function of  $n$  variables:  $g = f(x_1, x_2, \dots, x_n)$  around a point  $x_0 = (x_{10}, x_{20}, \dots, x_{n0})$ :

$$g = f(x_{10}, x_{20}, \dots, x_{n0}) + \frac{\partial f}{\partial x_1} \Big|_{x_0} (x_1 - x_{10}) + \dots + \frac{\partial f}{\partial x_n} \Big|_{x_0} (x_n - x_{n0}).$$

We evaluate the partial derivatives,  $\frac{\partial f}{\partial x_k} \Big|_{x_0}$  for all  $k = 1, \dots, n$ , which will lead to a constant, and by denoting  $\Delta x_k = x_k - x_{k0}$  for all  $k = 1, \dots, n$  a linear expression is obtained.

In our case we have two nonlinear equations in (2), the second and the fourth equation, and we put all the terms to the left hand side of the equation to obtain:

$$\begin{aligned} g_2(x_1, x_2, \dot{x}_2) &= \frac{x_1}{A_1} - \frac{s_1 \sqrt{2gx_2}}{A_1} - \dot{x}_2 \\ g_4(x_2, x_3, x_4, \dot{x}_4) &= \frac{s_1 \sqrt{2gx_2}}{A_2} - \frac{s_2 \sqrt{2gx_4}}{A_2} + \frac{x_3}{A_2} - \dot{x}_4. \end{aligned} \quad (13)$$

By applying the first order Taylor series approximation the following is obtained:

$$\begin{aligned} g_2(x_1, x_2, \dot{x}_2) &= \frac{1}{A_1} (x_1 - x_{10}) + \left( -\frac{s_1 \sqrt{2g}}{2A_1 \sqrt{x_2}} \right) \Big|_{x_0} (x_2 - x_{20}) - \dot{x}_2 \\ &= 0.04(x_1 - x_{10}) + (-0.39)(x_2 - x_{20}) - \dot{x}_2 \\ &\Rightarrow \dot{x}_2 = 0.04\Delta x_1 - 0.39\Delta x_2. \end{aligned} \quad (14)$$

Similarly, with  $g_4$  we obtain:

$$\dot{x}_4 = 0.61\Delta x_2 + 0.06\Delta x_3 - 1.2\Delta x_4. \quad (15)$$

The linear system is written in terms:

$$\Delta x = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} = \begin{bmatrix} F_1 - F_{10} \\ h_1 - h_{10} \\ F_2 - F_{20} \\ h_2 - h_{20} \end{bmatrix} = \begin{bmatrix} F_1 - 99.5 \\ h_1 - 5 \\ F_2 - 16.03 \\ h_3 - 3 \end{bmatrix}, \quad \Delta u = \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} = \begin{bmatrix} u_1 - 19.8 \\ u_2 - 1.6 \end{bmatrix}. \quad (16)$$

By using the parameter values in Table 1, we obtain the following linear system:

$$\Delta \dot{x} = A\Delta x + B\Delta u, \quad A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0.04 & -0.39 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0.62 & 0.06 & -1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix}. \quad (17)$$

Based on the eigenvalues of  $A$ ,  $\text{eig}(A) = [-2 \quad -0.39 \quad -1 \quad -1.2]$ , we see that the unforced system is globally asymptotically stable, since all the eigenvalues are on the left half plane, [1].

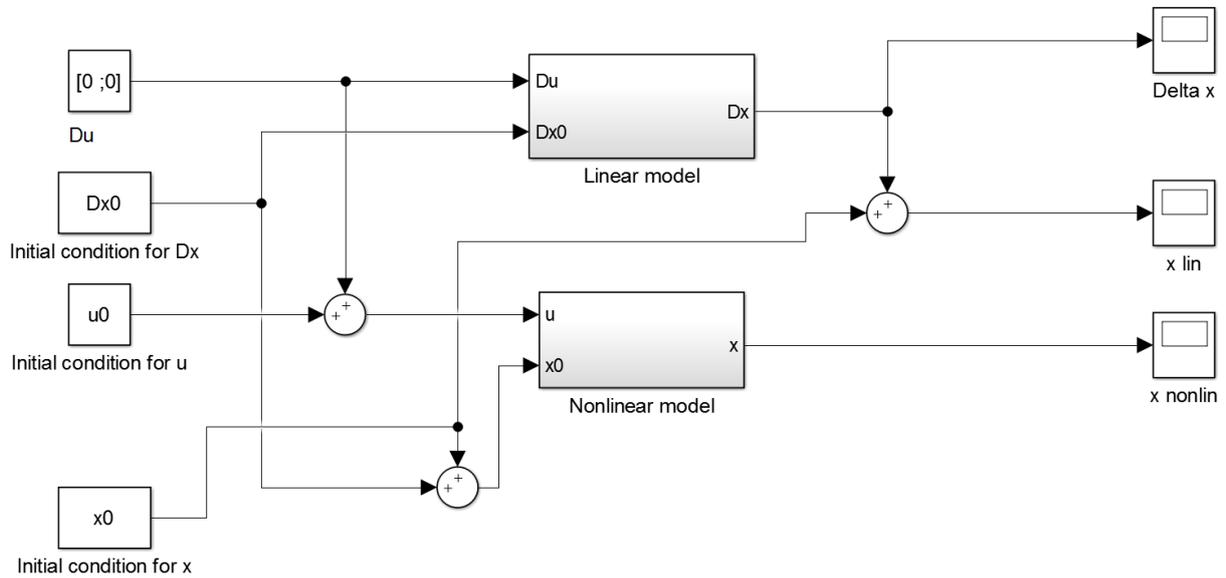


Figure 2: Open loop system, linear and nonlinear

For simulation consider the Simulink model described in Fig. 2. As it can be seen on Fig 2 the linear model defines the system in form of  $Dx$ ,  $Du$ , which refers to  $\Delta x$  and  $\Delta u$ . To compare correctly we need to add the linearization point,  $x_0$ . Then the initial condition for the nonlinear model is the linearization point,  $x_0$ , plus some deviation from that point, which is denoted with  $Dx0$ . The control input for the nonlinear system is  $u_0$  plus  $Du$ , and the control input for the linear system is just  $Du$ , because in the linear case we are already at the linearization point.

The nonlinear subsystem in Simulink contains the nonlinear equations described in (2) and an integrator. This can be seen on Fig. 3. Similar idea is applied for the linear model.

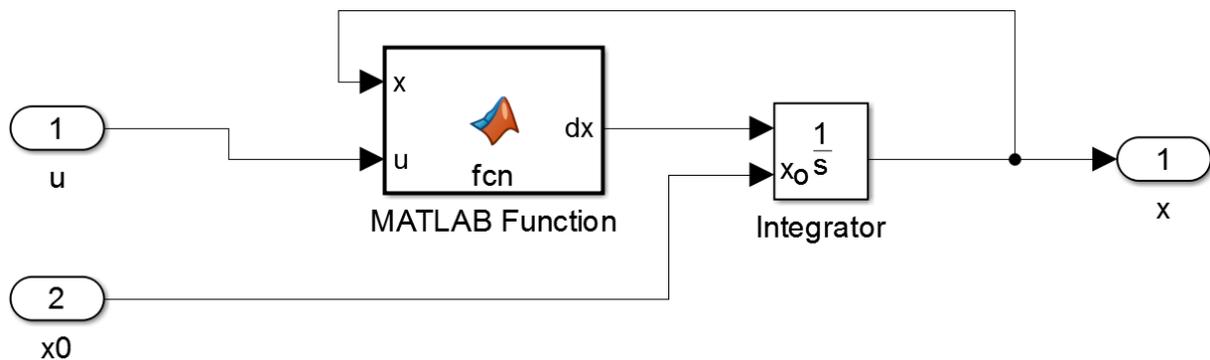


Figure 3: Nonlinear model

On Fig. 4 a simulation can be seen on the nonlinear system, with initial conditions  $\Delta x_0 = Dx0 = [10 \ 5 \ 6 \ 3]^T$  and the system is unforced, i.e.  $\Delta u = Du = [0 \ 0]^T$ . The difference between the linear and nonlinear system can be seen on Fig. 5. If we take a closer look on Fig. 5, we can see that the difference between the linear and nonlinear unforced model is relatively small, furthermore, for states  $x_1$  and  $x_3$  there is no difference. To explain this, let's look again to (2), from where we can see that both

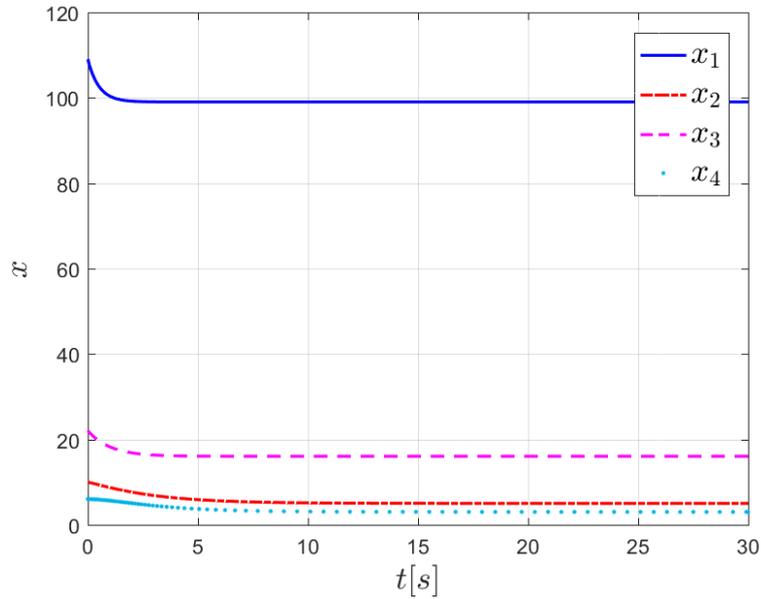


Figure 4: Nonlinear model simulation with:  $\Delta x_0 = [10 \ 5 \ 6 \ 3]^T$

$\dot{x}_1 = f_1(x_1, u_1)$  and  $\dot{x}_3 = f_3(x_3, u_2)$  are linear, and the dynamics of  $x_1$  and  $x_3$  are defined only using themselves and the inputs. So there is no difference in this two states between the linear and the nonlinear case.

Based on the eigenvalues of the linear model in (17) we saw that the unforced system is stable, so it must go back to the equilibrium point, which is can also be seen on Fig. 4.

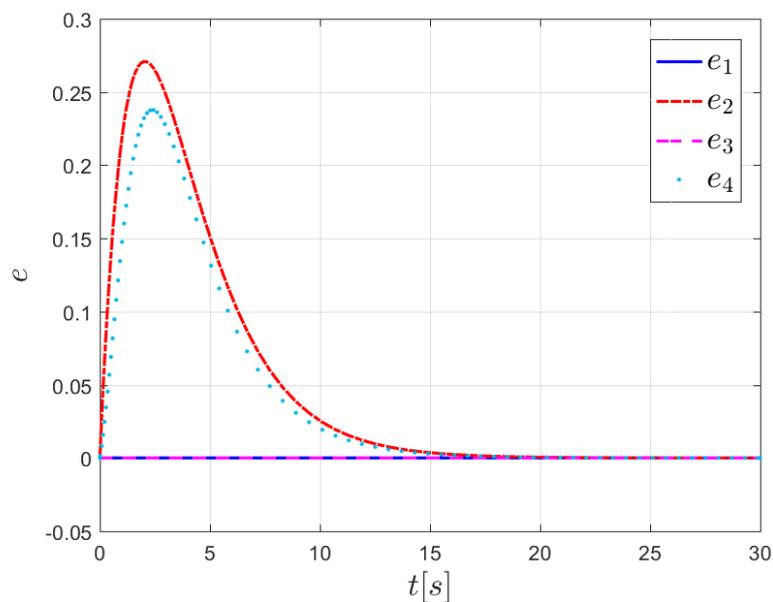


Figure 5: Difference between linear and nonlinear models

## 3 State-feedback control

### 3.1 Theoretical background

Consider the linear system:

$$\dot{x} = Ax + Bu,$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , and  $A, B$  are the system matrices. Consider the state feedback control law,  $u = -Kx$ , which leads to:

$$\dot{x} = Ax - BKx = (A - BK)x. \quad (18)$$

Based on [1] the closed loop system in (18) is stable if the eigenvalues of  $(A - BK)$  are negative. But for this, first we need to check if the pair  $(A, B)$  is controllable. If it is not fully state controllable, it means that there is no possible gain  $K$ , for which the system will be stable. We use the controllability matrix to check the controllability:

$$P_c = [B \ AB \ \dots \ A^{n-1}B]. \quad (19)$$

If the rank of  $P_c$  is equal with the number of states  $n$ , then the system is fully state controllable, and the gain  $K$  can be obtained. For a better understanding let us consider the following numerical example:

$$\dot{x} = Ax + Bu, \quad A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

We see that the unforced (zero input) system is unstable, since  $\text{eig}(A) = [1, -2]$ .

It can happen that the unforced system is stable, i.e. all the eigenvalues are negative, but we want to change some of the performances of the system, for instance to reduce the settling time or to reduce the overshoot, in these cases we can also use the state-feedback control law, with poles for which the desired performances are achieved.

The controllability matrix is  $P_c = [B \ AB] = \begin{bmatrix} 0 & 3 \\ 3 & -3 \end{bmatrix}$ , and  $\text{rank}(P_c) = 2$ , so the system is fully state controllable. We consider  $K = [k_1 \ k_2]$ , so

$$A - BK = \begin{bmatrix} 0 & 1 \\ 2 - 3k_1 & -3k_2 - 1 \end{bmatrix}.$$

To find the eigenvalues, we have characteristic equation:

$$\det(\lambda I - A + BK) = \lambda^2 + (3k_2 + 1)\lambda + 3k_1 - 2 = 0. \quad (20)$$

We impose that the desired closed loop eigenvalues are  $\lambda_{1,2} = [-1, -2]$ , so the desired characteristic equation is  $\lambda^2 + 3\lambda + 2 = 0$ . From the desired characteristic equation and (20), we obtain that  $3k_2 + 1 = 3$  and  $3k_1 - 2 = 2$ , which leads to  $k_1 = 1.33$ ,  $k_2 = 0.66$ .

Note that for higher order systems the calculation of the gain is not straight forward. Matlab can be used for these cases with function `place`, in the following form: `K = place(A, B, [eigdesired])`.

### 3.2 Tank system with control

We saw in the simulation that, the system described by (17) is stable, so it will go back to the equilibrium point. On Fig. 4 we can see that  $x_1$  and  $x_3$  are stabilizing relatively fast, within 3 seconds, but states  $x_2$  and  $x_4$  needs more time to stabilize. On Fig. 6 we zoom in to see more clearly what happens with  $x_2$  and  $x_4$ . As it can be seen it requires about 15 seconds to stabilize. We would like to obtain a faster stabilization, i.e. we want to reduce the settling time, that is why we design a state-feedback control. The eigenvalues of the  $A$  matrix are actually the poles of the system, and based on [2] we know that the system response is faster if the poles are further away in the negative direction from the imaginary axis. The controlled design techniques can be further explored from [2], but this is not the main focus of this laboratory guide.

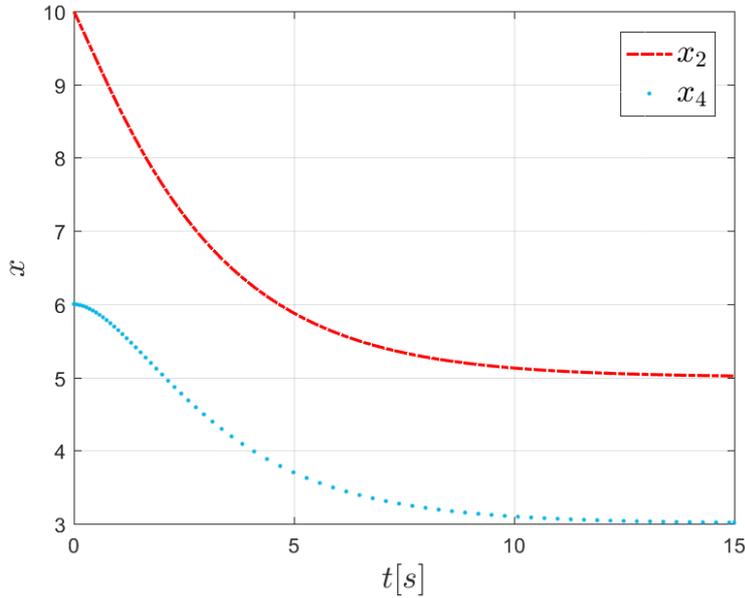


Figure 6: Nonlinear model  $x_2$  and  $x_4$  with:  $\Delta x_0 = [10 \ 5 \ 6 \ 3]^T$

The system (17) is fully state controllable since the rank of the controllability matrix is 4. We want a 3 times faster settling-time for the states  $x_2$  and  $x_4$  in (17), so we change the closed loop eigenvalues in these positions:  $\text{eig}_{des}(A - BK) = [-2 \ -1.2 \ -1 \ -3.6]$ . Since the states are depending on each other the obtained result might not necessarily be 3 times faster. We consider the state feedback control law:

$$\Delta u = -K\Delta x, \quad (21)$$

where  $K$  is the state-feedback gain. By using `place` function in Matlab, we obtain:

$$K = \begin{bmatrix} 0.1681 & 4.1941 & 0.0613 & -0.0063 \\ 0.1007 & 1.8614 & 0.1524 & -0.0053 \end{bmatrix}. \quad (22)$$

The linear closed-loop model, with control law (21) represented in Simulink can be seen on Fig. 7. As it was described in (21) in the control law the  $\Delta x$  is used and the obtained control input is  $\Delta u$ . Similar model can be obtained for the nonlinear model, but we need to pay attention, since the control law in (21) is defined with  $\Delta x$ , so from the output we need to subtract  $x_0$ . On the other hand, the computed

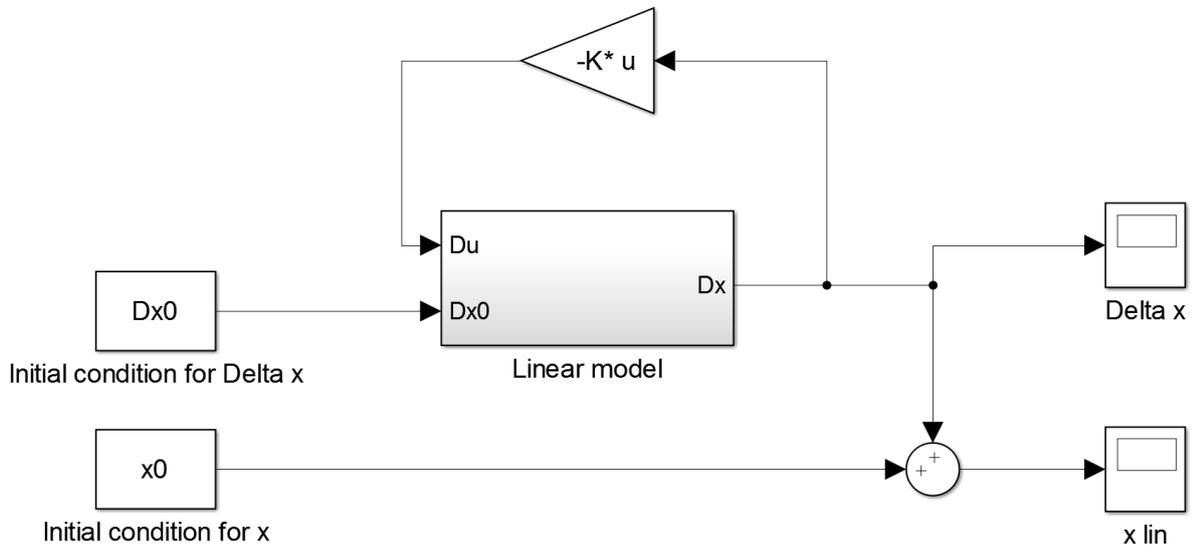


Figure 7: Linear closed-loop system

input is  $\Delta u$  and we need to add the equilibrium input  $u_0$ . The Simulink model can be seen on Fig 8. In both of the cases the same state feedback gain was used from (22). The obtained results for the linear

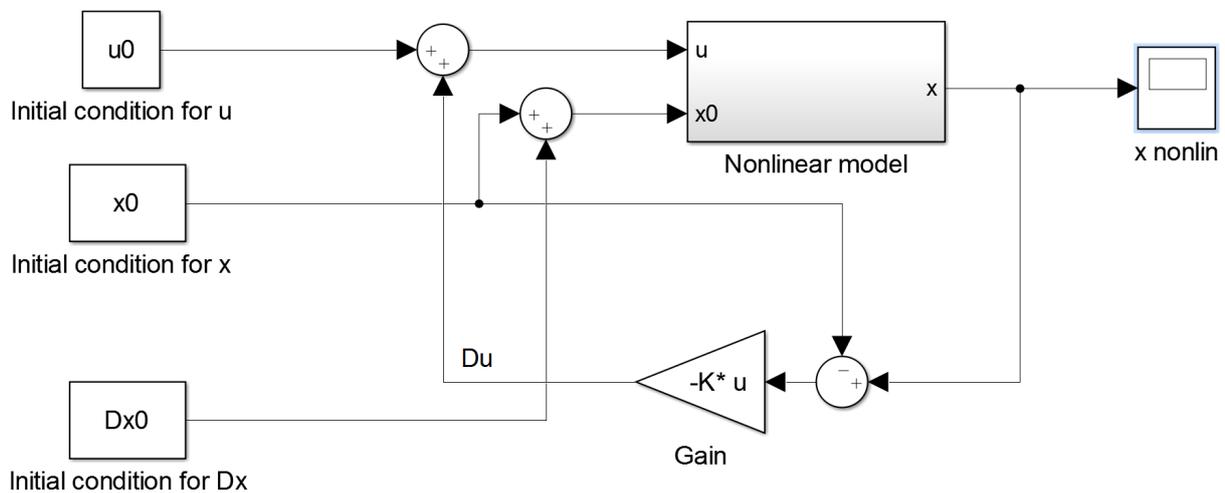


Figure 8: Nonlinear closed-loop system

system can be seen on Fig. 9; we used the same initial condition for  $\Delta x$  as in Fig. 4. We can see that the settling time for  $x_2$  and  $x_4$  are reduced, it is around 5 seconds now. A similar result is obtained with the same control law, but for the nonlinear model; we show only the two  $x_2$  and  $x_4$ , this can be seen on Fig. 10.

Based on the obtained results we obtained the desired performance, i.e. we reduced the settling time, and the results are acceptable on both linear and nonlinear models.

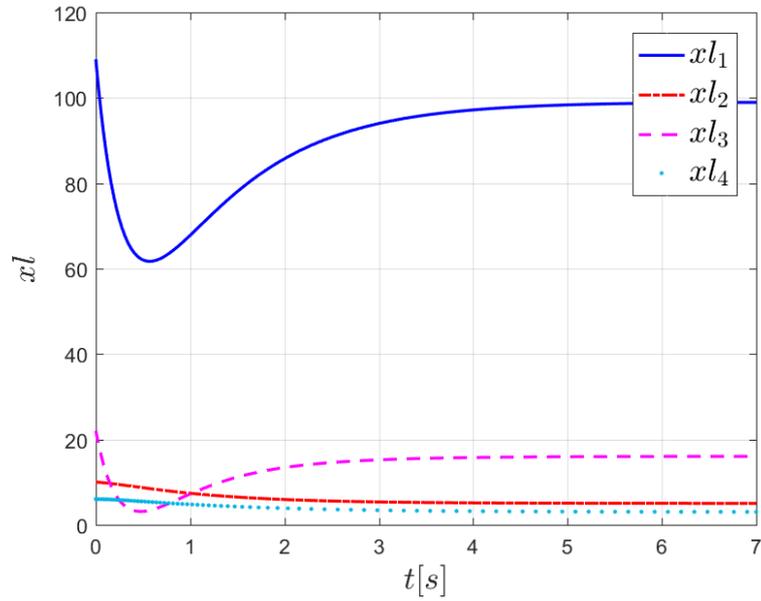


Figure 9: Closed loop linear model, with  $\Delta x_0 = [10 \ 5 \ 6 \ 3]^T$

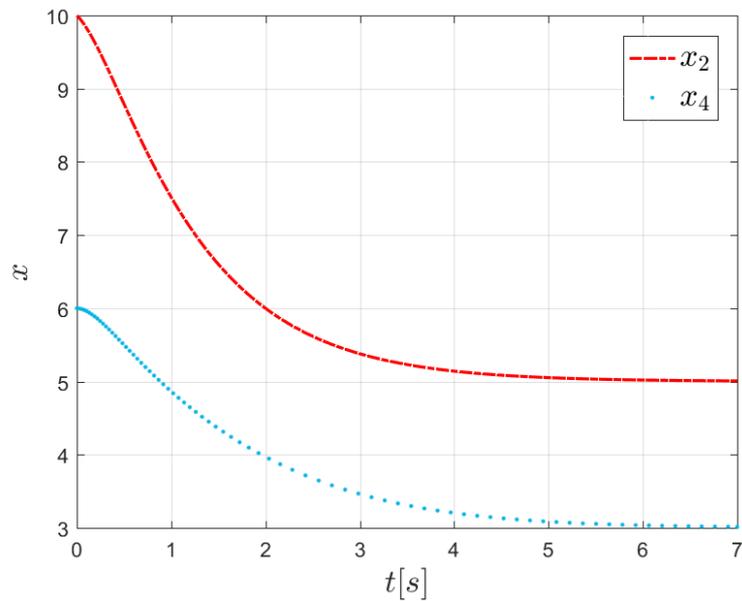


Figure 10: Closed loop nonlinear model,  $x_2$  and  $x_4$  with  $\Delta x_0 = [10 \ 5 \ 6 \ 3]^T$

## 4 Observer design

### 4.1 Theoretical background

Until this point it was assumed that all the states are measurable and available for control. In this section we consider the case where some of the states are unavailable and we need to estimate them, so we can

use them for control. Consider the linear system:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx,\end{aligned}\tag{23}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^r$ ,  $A$ ,  $B$  are the system matrices, and  $C$  is the output matrix. This means that what we can actually measure is the output  $y$ , and we have no access to the other states.

Let us consider the following observer:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\ \hat{y} &= C\hat{x},\end{aligned}\tag{24}$$

where  $\hat{x}$  is the estimate of  $x$ , and  $L$  is the observer gain. We denote  $e = x - \hat{x}$ , the difference between the true value and the estimated state vector. The observer is working well if this  $e$ , the error dynamics, is converging to 0, which means that the estimated state vector is converging to the true state vector. We can write the error dynamics:

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} = (Ax + Bu) - (A\hat{x} + Bu + L(y - \hat{y})) = A(x - \hat{x}) - L(y - \hat{y}) \\ &= Ae - L(Cx - C\hat{x}) = (A - LC)e.\end{aligned}\tag{25}$$

At this point, based on [1] we can say that the estimation error  $e$  is globally asymptotically stable, i.e. is converging to 0, if the matrix  $(A - LC)$  has negative eigenvalues. As a matter of notations the matrix which has negative eigenvalues is called a Hurwitz matrix.

In order to design the observer first we need to verify if the system is fully state observable. For this reason the observability matrix need to be calculated:

$$P_o = \begin{bmatrix} C \\ CA \\ \vdots \\ \vdots \\ CA^{n-1} \end{bmatrix}.\tag{26}$$

If the rank of the observability matrix is equal with the number of states, then the system is fully state observable, and an observer gain  $L$  can be calculated. Similarly as it is done in the state-feedback control design, we can use the `place` function in Matlab in the following way:

$$L^T = \text{place}(A^T, C^T, [\text{eigdesired}]).$$

## 4.2 Tank system with observer

For the example described in (2) we assume that the levels in the tanks are available:  $x_2$ ,  $x_4$ ; and we want to estimate the flow rates:  $x_1$ ,  $x_3$ . This idea makes sense also equipment wise, since the in the case of a real system we can easily measure the level in the tanks with some resistance based level sensors, but to measure flow rate, more sophisticated sensors are needed, which increase the price of the overall equipment. To avoid this, a linear observer can be used. Let us consider again the linear model (17) extended with the output matrix  $C$ :

$$\begin{aligned}\Delta\dot{x} &= A\Delta x + B\Delta u \\ \Delta y &= C\Delta x,\end{aligned}\tag{27}$$

and in order to consider just the levels the following output matrix is used:

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (28)$$

It is important to mention that the output of the linear model is defined with  $\Delta y$  so it will give us the relative output from the linearization point:  $\Delta y = y - y_0$ , where  $y_0 = Cx_0$ .

The following linear observer is considered:

$$\begin{aligned} \Delta \dot{\hat{x}} &= A\Delta \hat{x} + B\Delta u + L(\Delta y - \Delta \hat{y}) \\ \Delta \hat{y} &= C\Delta \hat{x}, \end{aligned} \quad (29)$$

where  $\Delta \hat{x}$  is the estimate of  $\Delta x$ ,  $L$  is the observer gain.

For the linear tank system in (17) the rank of the observability matrix is 4, so the linear model is fully state observable. We want a fast observation, or at least faster than the controller, otherwise because of a slow observer the overall system can become unstable, when we use the estimated states for the control. For this reason we consider the following desired eigenvalues:  $\text{eig}_{des}(A - LC) = [-2 \quad -1.2 \quad -1 \quad -3.6]$ . The obtained observer gain is the following:

$$L = \begin{bmatrix} 0.0192 & 0.0184 \\ 6.2007 & 0.9542 \\ 47.5293 & 82.5393 \\ 2.3811 & 4.8044 \end{bmatrix}. \quad (30)$$

In the following figures we show step by step how can we add the observer to the model. On Fig. 11 we can see the open loop linear model with and the observer. The notations have the following meaning:

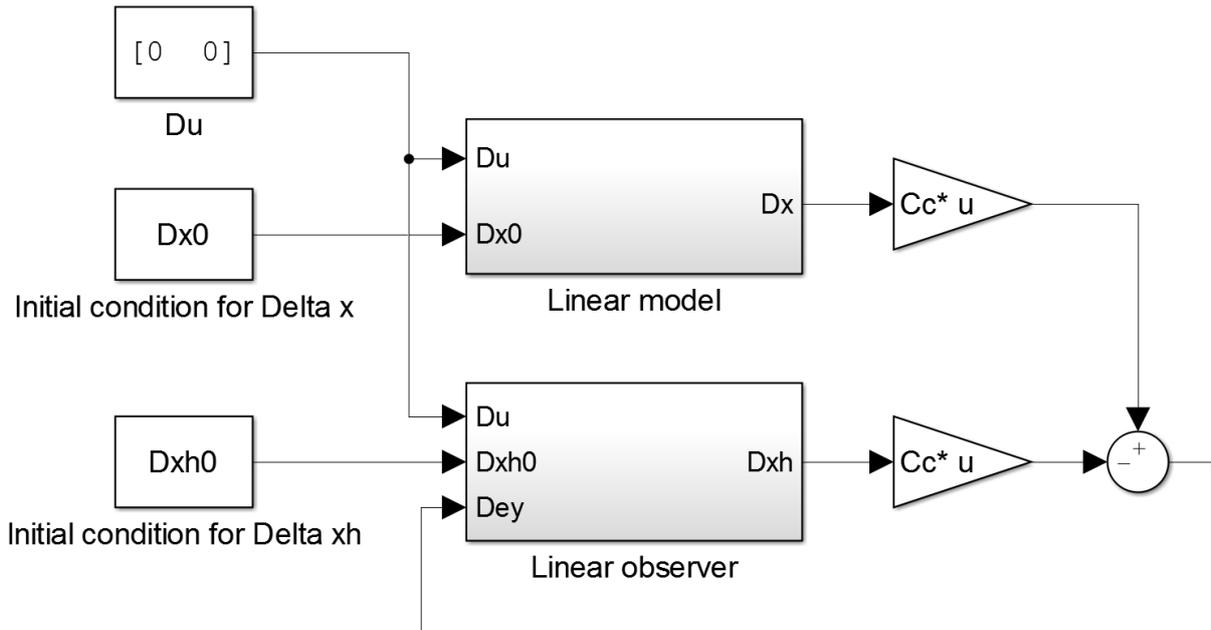


Figure 11: Open-loop model with linear observer

$Dxh = \Delta \hat{x}$ ,  $Dxh0 = \Delta \hat{x}_0$ ,  $Cc = C$ , and  $Dey = \Delta y - \Delta \hat{y}$ . On Fig. 12 the closed loop model can be

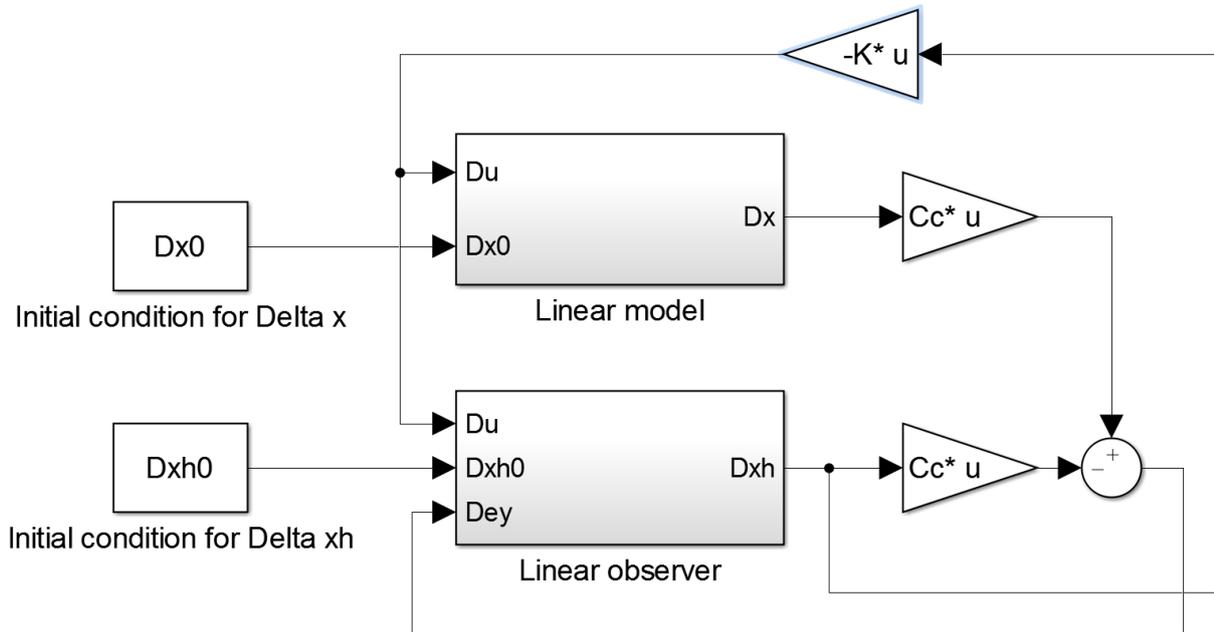


Figure 12: Closed-loop linear model with linear observer

seen together with the observer. Since now we do not have direct access to the true states of the system,  $\Delta x$ , we consider the estimated states,  $\Delta \hat{x}$ , to calculate the control input. We considered the following initial conditions  $Dx0 = [10 \ 5 \ 6 \ 3]^T$  and  $Dxh0 = [0 \ 0 \ 0 \ 0]^T$ . As it can be seen on Fig. 13 the error dynamics  $e = Dx - Dxh$  is converging to 0. It is important to mention that, the obtained results on

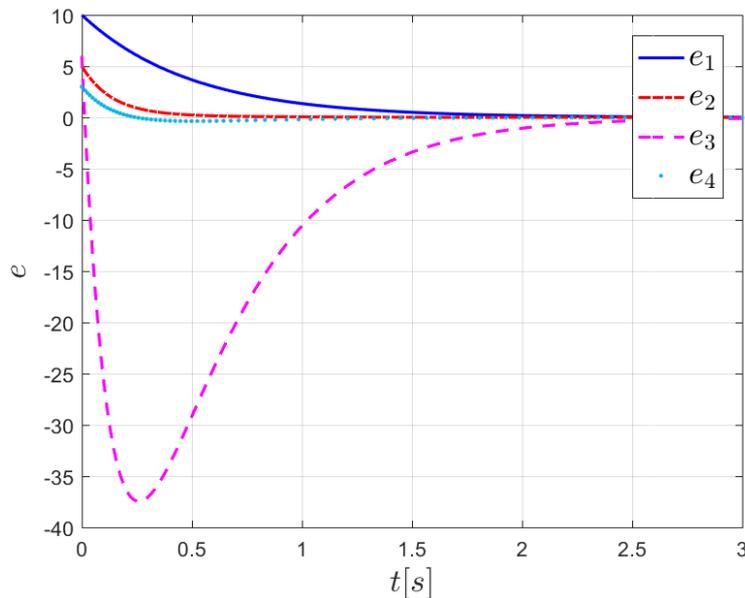


Figure 13: Estimation error with  $\Delta x_0 = [10 \ 5 \ 6 \ 3]^T$  and  $\Delta \hat{x}_0 = [0 \ 0 \ 0 \ 0]^T$

Fig. 12 are for  $\Delta x$  and  $\Delta \hat{x}$ , and in order to find the true values we need to add the equilibrium point:

$$x_{linear} = x_0 + \Delta x.$$

The observer can be used also for the nonlinear model, but we need to take into consideration the equilibrium point; the Simulink model corresponding to this is represented on Fig. 14.

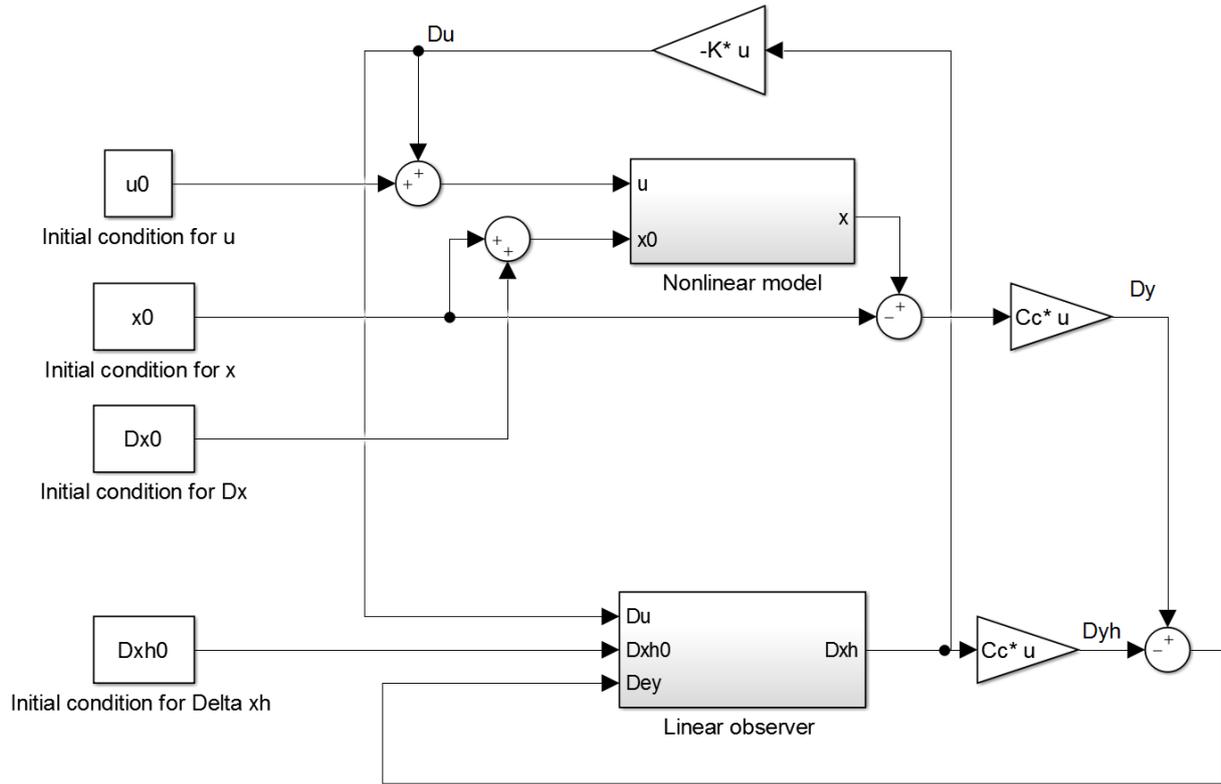


Figure 14: Closed-loop nonlinear model with linear observer

#### 4.2.1 Possible improvements

In most of the cases the linear observer is working well if we are close to the linearization point, but a detailed stability analysis is needed in order to define the exact range until which the linear observer is working. A possible improvement to this problem is to use the nonlinear dynamics also in the observer. Let us consider the following observer:

$$\dot{\hat{x}} = f(\hat{x}, u) + L(y - \hat{y}) \quad (31)$$

Even in some cases we can examine more the nonlinearities and we can separate them into two: one which depends only on the output and the other one contains nonlinearities depending on the unmeasured states. If we take a closer look to (2), we can see that the nonlinearities are appearing in  $f_2(x_1, x_2)$  and  $f_4(x_2, x_3, x_4)$ , and they depend only on  $x_2$  and  $x_4$ , which corresponds to the levels in the tanks, and they are the measured outputs. We can rewrite (2) in the following form:

$$\begin{aligned} \dot{x} &= A_{lin}x + f_{nl}(y) + B_{lin}u \\ y &= Cx, \end{aligned} \quad (32)$$

where

$$f_{nl}(y) = \begin{bmatrix} 0 \\ \frac{s_1\sqrt{2g y_1}}{A_1} \\ 0 \\ \frac{s_1\sqrt{2g y_1}}{A_2} - \frac{s_2\sqrt{2g y_2}}{A_2} \end{bmatrix}. \quad (33)$$

We consider the following observer:

$$\begin{aligned} \dot{\hat{x}} &= A_{lin}\hat{x} + f_{nl}(y) + B_{lin}u + L(y - \hat{y}) \\ \hat{y} &= C\hat{x}. \end{aligned} \quad (34)$$

Using this observer we can see that the error dynamics is linear and the error dynamics will be globally asymptotically stable.

## 5 Discrete-time modelling, control and estimation

In many real-life application a digital controller and observer is used, and for this reason in this section we convert all the previously explained elements in their discrete-time version.

### 5.1 Discrete-time modelling

There are many ways to obtain a discrete-time model, from which we highlight the Euler discretization. The basic idea is coming from the approximation of the derivation in the following way:

$$\dot{x} \simeq \frac{x(k+1) - x(k)}{T_s}. \quad (35)$$

It means that the derivative can be approximated with the difference between the state values at point  $k+1$  and  $k$ , divided by the sampling time,  $T_s$ . If the sampling time is small enough, then we have a good approximation. Lets see what happens with the linear model, if we use this approximation:

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow \frac{x(k+1) - x(k)}{T_s} = Ax(k) + Bu(k). \quad (36)$$

Basically the continuous-time is converted into a discrete-time model. In addition to this we can further simplify the model and express just  $x(k+1)$ :

$$x(k+1) - x(k) = T_s Ax(k) + T_s Bu(k) \Rightarrow x(k+1) = (I + T_s A)x(k) + T_s Bu(k). \quad (37)$$

We can define the new discrete-time state matrices as  $A_d = I + T_s A$ , and  $B_d = T_s B$ ; we have the following model:

$$x(k+1) = A_d x(k) + B_d u(k) \quad (38)$$

The nonlinear model can be converted using the same idea:

$$\begin{aligned} \dot{x}(t) = f(x(t), u(t)) &\Rightarrow \frac{x(k+1) - x(k)}{T_s} = f(x(k), u(k)) \\ &\Rightarrow x(k+1) = x(k) + T_s f(x(k), u(k)) \end{aligned} \quad (39)$$

### 5.2 Discrete-time Tank model

In the discrete-time case we do need to use a differential equation solver, since both (38) and (39) defines the states in the next time step. For this reason we define the iterations in a pseudo-code instead of a Simulink model.

The discrete-time linear model is based on (17), and has the following form:

$$\Delta x(k+1) = A_d \Delta x(k) + B_d \Delta u(k), \quad (40)$$

where

$$A_d = I + T_s \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0.04 & -0.39 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0.62 & 0.06 & -1.2 \end{bmatrix}, \quad B_d = T_s \begin{bmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 10 \\ 0 & 0 \end{bmatrix}. \quad (41)$$

## References

- [1] H. K. Khalil, *Nonlinear Systems, Third edition*, P. Education, Ed. Prentice Hall, Upper Saddle River, NJ 07458, 2000.
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