Local quadratic and nonquadratic stabilization of discrete-time TS fuzzy systems

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***Abstract*—This paper proposes conditions for the local stabilization of discrete-time Takagi-Sugeno fuzzy systems, for which all existing conditions consider only global stabilization. The conditions are developed first considering a quadratic Lyapunov function and are then extended for non-quadratic Lyapunov functions. An estimate of the region of attraction is also obtained. The conditions are illustrated and discussed on a numerical example.**

I. Introduction

In the last two decades, Takagi-Sugeno (TS) fuzzy models [1] have attracted considerable interest in the automatic stability analysis or controller design of nonlinear systems. TS models represent the nonlinear systems considered as convex combinations of local linear models, and are able to exactly represent the system in a compact set of the state-space.

For the analysis and synthesis of TS models the direct Lyapunov approach has been employed, using initially quadratic Lyapunov functions [2]–[4], then piecewise continuous Lyapunov functions [5], [6], and recently, nonquadratic Lyapunov functions [7]–[9]. The conditions for stability analysis or controller and observer design are developed in general in form of linear matrix inequalities (LMIs) that can be solved using available convex optimization methods.

Existing conditions based on quadratic Lyapunov functions, developed both for analysis and synthesis, imply the global asymptotic stability of the (closed-loop) TS system. This in fact leads to the convergence of any trajectory starting in the largest Lyapunov level set included in the considered compact set of the state-space. For continuous-time systems this changed with the introduction of nonquadratic Lyapunov functions. In this case, the developments involve the derivatives of the membership functions, which provides significant challenges in developing non-conservative LMI conditions. Due to the appearance of the derivatives, which needed to be bounded, local stabilization results have been obtained, with the domain given by the bounds on the derivatives [9]–[11] of the membership functions. These bounds have usually been expressed as bounds on the states.

In the discrete-time case, non-quadratic Lyapunov functions have shown a real improvement [7], [12]–[14] for developing global stability and design conditions. For such systems, the variation of the Lyapunov function does not involve any derivatives and thus further bounds to be included in the conditions. Recently, by using Polya's theorem [15], [16] asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability in the sense of a chosen quadratic or nonquadratic Lyapunov function. The work in [17] gave ANS stability conditions for both membership function-dependent model and membership function-dependent Lyapunov matrix. A shortcoming of these results is that the number of LMIs that have to be solved increase quickly, leading to numerical intractability [18]. Furthermore, all these results give conditions for global stability or stabilization, i.e., if an equilibrium point is not globally stable or cannot be globally stabilized, no conclusion can be drawn.

Keeping in mind that the TS model is actually a representation of a nonlinear system, this system may have several equilibrium points due to which possibly only local stabilization can be achieved. Therefore, in this paper, we consider the problem of local stabilization of discrete-time TS models and estimating a domain of attraction of the equilibrium point. The structure of the paper is as follows. Section II presents the notations used in this paper and motivates our work through a simple example. Section III develops the proposed conditions for local stabilization, using a common quadratic and a nonquadratic Lyapunov function, respectively. The developed conditions are discussed and illustrated on a numerical example in the same section. Section IV concludes the paper.

II. Notation and preliminaries

In this paper we develop sufficient conditions for local stabilization of nonlinear discrete-time systems represented by Takagi-Sugeno (TS) fuzzy models. Thus, we consider systems of the form

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{r} h_i(\boldsymbol{z}_i(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k))$$

or, in shorthand notation

$$\boldsymbol{x}(k+1) = A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \tag{1}$$

where $\boldsymbol{x}$ denotes the state vector, $\boldsymbol{u}$ is the input vector, r is the number of rules, $\boldsymbol{z}$ is the scheduling vector, h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, and A_i, B_i, $i = 1, 2, \ldots, r$ are the local models. To motivate the research presented hereafter, consider the following example.

Example 1. *Consider the nonlinear system:*

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= 3x_1(k) + 10x_2(k) + u(k) \end{aligned} \tag{2}$$

with $x_1(k) \in [-a, a]$, $a > 0$ being a parameter. It can be easily seen that (2) *can only be stabilized at zero if $x_1(0) \in (-1, 1)$.*

The nonlinearity is x_1^2 and using the sector nonlinearity approach [19] on the domain $x_1(k) \in [-a, a]$, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k))A_1\boldsymbol{x} + h_2(x_1(k))A_2\boldsymbol{x} + Bu(k)$$

with $h_1(x_1) = \frac{a - x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -a & 0 \\ 3 & 10 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 3 & 10 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $a < 1$, say, $a = 0.9$, the TS model can be stabilized e.g., using controller design conditions developed based on a common quadratic Lyapunov function.

If the sector nonlinearity approach is applied for $a > 1$, without including further conditions, no conclusion can be drawn regarding stabilization of the TS model. A condition that leads to the feasibility of the associated LMI problem and thus makes it possible to draw some conclusion on the local stability of the closed-loop system is, e.g., $x_1^2(k) \geq 0.9x_1^2(k+1)$. However, the question on how to obtain this condition and its exact interpretation remains open.

In what follows, 0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry in matrices and the transpose of the left-hand side in inline expressions. The superscript $-T$ denotes the transpose of the inverse, and the subscript $z + m$ (as in A_{z+m}) stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., at $\boldsymbol{z}(k + m)$. We will also make use of the following results:

Lemma 1. *[20] Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{R} \in \mathbb{R}^{m \times n_x}$ such that rank$(\mathcal{R}) < n_x$. The two following expressions are equivalent:*

1) $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, \mathcal{R}\boldsymbol{x} = 0\}$
2) $\exists \mathcal{M} \in \mathbb{R}^{m \times n_x}$ *such that* $Q + \mathcal{M}\mathcal{R} + \mathcal{R}^T\mathcal{M}^T < 0$

Proposition 1. *(Congruence) Given a matrix $P = P^T$ and a full column rank matrix Q it holds that*

$$P > 0 \;\Rightarrow\; QPQ^T > 0$$

Proposition 2. *Let A and B be matrices of appropriate dimensions and ranks, with $B = B^T \geq 0$. Then*

$$(A - B)^T B^{-1}(A - B) \geq 0 \Leftrightarrow A^T B^{-1} A \geq A + A^T - B$$

Proposition 3. *[21] (Schur complement) Consider a matrix $M = M^T = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$, with M_{11} and M_{22} being square matrices. Then*

$$M < 0 \Leftrightarrow \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}$$

Proposition 4. *(S-procedure) Consider matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, such that $\boldsymbol{x}^T F_i \boldsymbol{x} \geq 0$, $i = 1, \ldots, p$, and the quadratic inequality condition*

$$\boldsymbol{x}^T F_0 \boldsymbol{x} > 0 \tag{3}$$

$\boldsymbol{x} \neq 0$. A sufficient condition for (3) *to hold is: there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, such that*

$$F_0 - \sum_{i=1}^{p} \tau_i F_i > 0$$

Analysis and design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{j=1}^{r} \sum_{k=1}^{r} h_j(\boldsymbol{z}(k)) h_k(\boldsymbol{z}(k)) \Gamma_{j,k} \boldsymbol{x} < 0 \tag{4}$$

where $\Gamma_{j,k}$, $j, k = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 2. *[22] The double-sum* (4) *is negative, if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r,\, i < j \end{aligned} \tag{5}$$

Lemma 3. *[23] The double-sum* (4) *is negative, if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r,\, i \neq j \end{aligned} \tag{6}$$

III. Local stabilization

In what follows, we will develop conditions that ensure local stabilization of discrete-time TS models. We first consider a common quadratic Lyapunov function, then extend the results to nonquadratic Lyapunov functions and non-PDC controllers.

A. Local quadratic stabilization

In this section, consider the controller design problem for the system (1). The controller used is

$$\boldsymbol{u}(k) = -F_z P^{-1} \boldsymbol{x}(k) \tag{7}$$

The closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (A_z - B_z F_z P^{-1})\boldsymbol{x}(k) \tag{8}$$

Our goal is to develop conditions that ensure that this system has a locally asymptotically stable equilibrium point in $\boldsymbol{x} = 0$ and determine a region of attraction. For determining the stabilization conditions, first a quadratic Lyapunov function of the form $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k) P^{-1} \boldsymbol{x}(k)$ will be used.

Furthermore, let us assume that there exists a matrix $R = R^T$ and a domain $\mathcal{D}_R$ in which

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \tag{9}$$

holds.

Then, the following result can be stated.

Theorem 1. *The closed-loop system* (8) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, F_i, $i = 1, 2, \ldots, r$ and $W = W^T$ so that*

$$\begin{pmatrix} -P & (*) \\ A_z P - B_z F_z & -P \end{pmatrix} + W < 0$$

holds. Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$, with R given by $R = \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} W \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix}$.

Proof. The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} A_z - B_z F_z P^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathcal{D}_R$, (9) holds. Thus, in this domain, (8) is locally asymptotically stable, if there exists $\mathcal{M}$ so that

$$\mathcal{M} \begin{pmatrix} A_z - B_z F_z P^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -P^{-1} & 0 \\ 0 & P \end{pmatrix} + R < 0$$

Choosing

$$\mathcal{M} = \begin{pmatrix} 0 \\ P^{-1} \end{pmatrix}$$

and congruence with

$$\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$$

leads to

$$\begin{pmatrix} -P & (*) \\ A_z P - B_z F_z & -P \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} R \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} < 0$$

Denoting $W = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} R \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$ we obtain the conditions in Theorem 1. Since the condition (9) holds only in the domain $\mathcal{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathcal{D}_R$. $\square$

Sufficient LMI conditions can easily be derived using Lemmas 3 or 2, as follows.

Corollary 1. *The closed-loop system* (8) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, F_i, $i = 1, 2, \ldots, r$ and $W = W^T$, so that* (5) *or* (6) *hold, with*

$$\Gamma_{i,j} = \begin{pmatrix} -P & (*) \\ A_i P - B_i F_j & -P \end{pmatrix} + W < 0$$

Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$, given by

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} W \times \begin{pmatrix} P^{-1} & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0$$

It should be noted that although Theorem 1 and Corollary 1 only establish convergence of the trajectories that start in $\mathcal{D}_S$, actually, any trajectory that eventually converges to $\mathcal{D}_S$ will converge to zero. Regarding the structure of W and, consequently, of R, several possibilities can be chosen, which will reflect different relations between consecutive states. To illustrate the use of the above conditions, and the effect that the structure of W will have on the result, consider the following example.

Example 2. *Consider the nonlinear system*

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= x_1(k) + 0.5 x_2(k) + u(k) \end{aligned} \tag{10}$$

with $x_1(k) \in [-2, 2]$, $a > 0$ being a parameter. It can be easily seen that (10) *can only be stabilized if $x_1(k) \in (-1, 1)$.*

Using the sector nonlinearity approach, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k)) A_1 \boldsymbol{x} + h_2(x_1(k)) A_2 \boldsymbol{x} + B u(k)$$

with $h_1(x_1) = \frac{a - x_1(k)}{4}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -2 & 0 \\ 1 & 0.5 \end{pmatrix}$, $A_2 = \begin{pmatrix} 2 & 0 \\ 1 & 0.5 \end{pmatrix}$, and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since the local models are not controllable, classical conditions for controller design fail. In what follows, our goal is to (locally) stabilize the system. For this, several structures of the matrix W are tested:

O_1: *full W. The results are presented in Figure 1(a). The resulting matrix R is*

$$R = 10^5 \begin{pmatrix} -1.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1.5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The resulting set $\mathcal{D}_S = 0$.

O_2: $W = \begin{pmatrix} W_1 & 0 \\ 0 & -I \end{pmatrix}$. *The results are presented in Figure 1(b). The resulting matrix R is*

$$R = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0.006 & 0 & 0 \\ 0 & 0 & -41.79 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The domain has been significantly increased.

O_3: $W = \begin{pmatrix} W_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ giving the results in Figure 1(c). The resulting matrix R is

$$R = \begin{pmatrix} 0.0007 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.016 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In this case, the resulting domain is reduced wrt. to the previous one due to the shape of the Lyapunov level set.

For all the cases above, the trace of R has been maximized. As can be seen, the recovered region that is proven stable is actually much smaller than the region containing those initial points that actually converge to zero using the computed control law. Consequently, the trajectories staring in these points will eventually converge also to $\mathcal{D}_R$. In what follows, consider the block-diagonal W above and verify those points that converge to $\mathcal{D}_R$ in Figure 1(b) in one and two steps, respectively. The results are shown in Figures 2(a) and 2(b), respectively.

As can be seen, a significant improvement of the domain is obtained. In fact, considering $\bigcup_{i=1}^{\beta} \mathcal{D}_{R_i}$, where $\mathcal{D}_{R_i}$ denotes the domain of those states whose trajectory will arrive in $\mathcal{D}_R$ in i steps and with β a relatively small, finite value, can improve the result.

Remark: Using $\mathcal{M} = [M_{1z}, M_{2z}]^T$ instead of $\mathcal{M} = [0, M_z]^T$ may improve the result. At this point, in order to reduce the number of variables involved and to be in line with results in the literature, we use $\mathcal{M} = [0, M_z]^T$.

A different way to improve the results is by developing conditions based on non-quadratic Lyapunov function.

B. Local non-quadratic stabilization

In this section, consider the controller design problem for the system (1), repeated here for convenience

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k) + B_z \boldsymbol{u}(k) \tag{11}$$

The controller used is of the form

$$\boldsymbol{u}(k) = -F_z H_z^{-1} \boldsymbol{x}(k) \tag{12}$$

and the closed-loop system can be expressed as

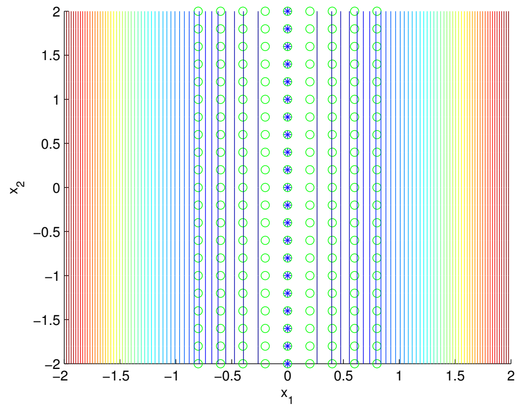
$$\boldsymbol{x}(k+1) = (A_z - B_z F_z H_z^{-1}) \boldsymbol{x}(k) \tag{13}$$

In what follows, two Lyapunov functions will be considered, similarly to the results in [24]:

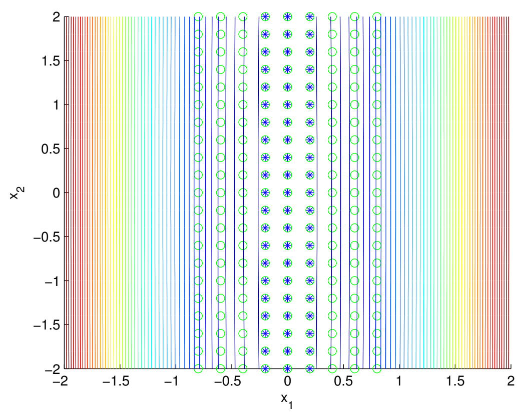
- Case 1: $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k) H_z^{-T} P_z H_z^{-1} \boldsymbol{x}(k)$
- Case 2: $V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k) P_z \boldsymbol{x}(k)$

Let us first consider the Lyapunov function in the Case 1 above, together with a domain $\mathcal{D}_R$ defined as

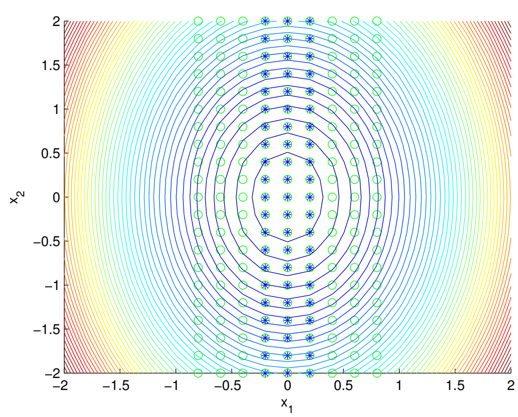
$$\begin{aligned} &\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} H_z^{-T} & 0 \\ 0 & H_{z+}^{-T} \end{pmatrix} R \\ &\times \begin{pmatrix} H_z^{-1} & 0 \\ 0 & H_{z+}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \end{aligned} \tag{14}$$



(a) Results using a full R.



(b) Results using the R in O_2.



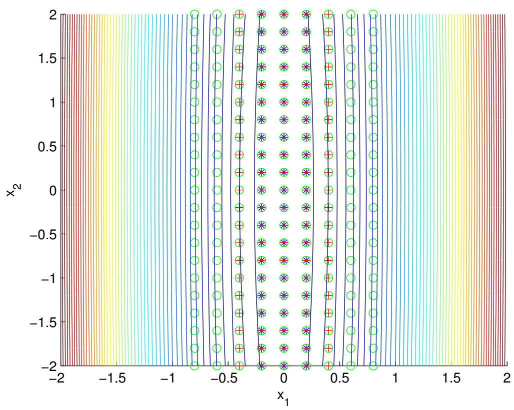
(c) Results using the specific R in O_3.

Fig. 1. Results for Example 2: $\mathcal{D}_R$ (blue $*$), the Lyapunov level sets, and the points that will actually converge with the computed control (green o)

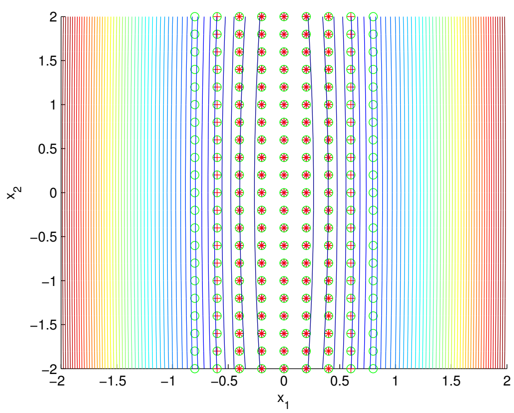
Then, the following result can be established:

Theorem 2. *The closed-loop system* (13) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, F_i, H_i $i = 1, 2, \ldots, r$ and R so that*

$$\begin{pmatrix} -P_z & (*) \\ A_z H_z - B_z F_z & -H_{z+} - H_{z+}^T + P_{z+} \end{pmatrix} + R < 0$$



(a) 0 and 1 step.



(b) 0,1 and 2 steps.

Fig. 2. Trajectories that converge to $\mathcal{D}_R$ for Example 2.

holds. Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$.

Proof. The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -H_z^T P_z H_z^T & 0 \\ 0 & H_{z+}^T P_{z+} H_{z+}^T \end{pmatrix} \times \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathcal{D}_R$, (14) holds. Thus, in this domain, (13) is locally asymptotically stable, if there exists $\mathcal{M}$ so that

$$\mathcal{M} \begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -H_z^T P_z H_z^T & 0 \\ 0 & H_{z+}^T P_{z+} H_{z+}^T \end{pmatrix} + \begin{pmatrix} H_z^{-T} & 0 \\ 0 & H_{z+}^{-T} \end{pmatrix} R \begin{pmatrix} H_z^{-1} & 0 \\ 0 & H_{z+}^{-1} \end{pmatrix} < 0$$

Choosing $\mathcal{M} = \begin{pmatrix} 0 \\ H_{z+}^{-T} \end{pmatrix}$ and congruence with $\begin{pmatrix} H_z^T & 0 \\ 0 & H_{z+}^T \end{pmatrix}$ leads to the condition of Theorem 2. Furthermore, since the condition (14) holds only in the domain $\mathcal{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathcal{D}_R$. □

Let us now consider the Lyapunov function in Case 2, together with a domain $\mathcal{D}_R$ defined as

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} H_z^{-T} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} R \times \begin{pmatrix} H_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} > 0 \tag{15}$$

For this case, the following result can be established:

Theorem 3. *The closed-loop system* (13) *is locally asymptotically stable if there exist matrices $P = P^T > 0$, F_i, H_i $i = 1, 2, \ldots, r$ and R so that*

$$\begin{pmatrix} -H_z - H_z^T + P_z & (*) \\ A_z H_z - B_z F_z & -P_{z+} \end{pmatrix} + R < 0$$

holds. Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$.

Proof. The difference in the Lyapunov function is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

and the closed-loop system can be expressed as

$$\begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} = 0$$

Furthermore, in the domain $\mathcal{D}_R$, (15) holds. Thus, in this domain, (13) is locally asymptotically stable, if there exists $\mathcal{M}$ so that

$$\mathcal{M} \begin{pmatrix} A_z - B_z F_z H_z^{-1} & -I \end{pmatrix} + (*) + \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_{z+} \end{pmatrix} + \begin{pmatrix} H_z^{-T} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} R \begin{pmatrix} H_z^{-1} & 0 \\ 0 & P_{z+}^{-1} \end{pmatrix} < 0$$

Choosing $\mathcal{M} = \begin{pmatrix} 0 \\ P_{z+}^{-1} \end{pmatrix}$ and congruence with

$$\begin{pmatrix} H_z^T & 0 \\ 0 & P_{z+} \end{pmatrix}$$

leads to

$$\begin{pmatrix} -H_z^T P_z^{-1} H_z & (*) \\ A_z H_z - B_z F_z & -P_{z+} \end{pmatrix} + R < 0$$

Applying Property 2 gives the condition of Theorem 3. Furthermore, since the condition (14) holds only in the domain $\mathcal{D}_R$, the region of attraction includes the largest Lyapunov level set contained in $\mathcal{D}_R$. □

Remark: Note that the conditions expressed in Theorems 2 and 3 above are not equivalent and do not include each other (see also [24]). Depending on the system considered one or the other may give less conservative results.

Similarly to the quadratic case, several possibilities can be chosen for W, such as diagonal, block-diagonal, a full one or any other structure. For instance, considering Example 2, choosing a diagonal W, and using the conditions of Theorem 2, the resulting matrix W is $W = \text{diag}\{22.75, 29.67, -116.25, 3.47\}$ and the estimated domain of attraction is graphically illustrated in Figure 3. On the other hand, the result obtained using Theorem 3 are not better than those obtained with a quadratic Lyapunov function.

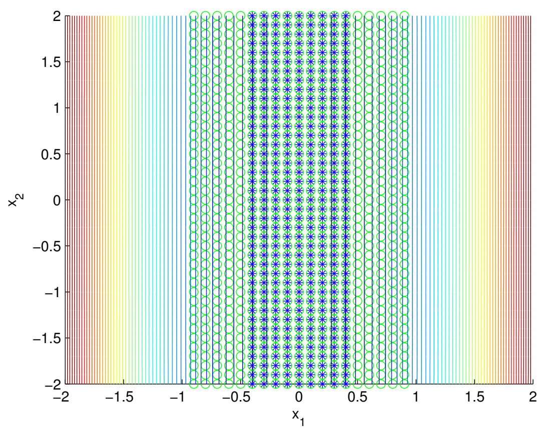


Fig. 3. Domain recovered with the conditions of Theorem 2 for Example 2.

IV. Conclusion

This paper proposed conditions for local stabilization of a TS fuzzy model. First a common quadratic Lyapunov function has been used and then the results have been extended to non-quadratic Lyapunov functions. The main idea was to combine the search for the control gains with the maximization of the domain of attraction using an LMI formalism. The developed conditions have been illustrated on a numerical example. In our future work, we will investigate how the structure of the weighting matrix W should be chosen and we will apply the proposed conditions in practice.

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