Controller design for discrete-time descriptor models: a systematic LMI approach

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Abstract—**The present paper provides a systematic way to generalize controller design for discrete-time nonlinear descriptor models. The controller synthesis is done using a Takagi-Sugeno representation of the descriptor models and linear matrix inequalities. We propose approaches to exploit the discrete-time nature of the problem by using delayed Lyapunov functions which allow delayed controller gains. In addition, the well-known Finsler's Lemma is used to avoid the explicit substitution of the closed-loop dynamics. Examples are provided to show the effectiveness of the proposed results.**

Index Terms—**Controller design, linear matrix inequality, nonlinear descriptor model, non-quadratic Lyapunov function.**

I. Introduction

A large class of nonlinear models can be exactly represented by Takagi-Sugeno (TS) models via the sector nonlinearity approach [1]. When this methodology is used, the designed controller/observer applies directly to the nonlinear system. The TS model is a collection of local linear models combined by nonlinear membership functions (MFs); these MFs share the convex sum property [2]. The TS model facilitates analysis via the direct Lyapunov method and linear matrix inequalities (LMIs). The interest of casting conditions in LMI terms is because they can be efficiently solved via convex optimization techniques [3], [4]. Nevertheless, the TS-LMI framework presents an important shortcoming: when using the sector nonlinearity approach, the number of rules (linear models) is exponentially related to the number of non-constant terms in the original nonlinear model, thus possibly turning the problem computationally intractable, especially when mechanical systems are under study.

Often, mechanical systems can be expressed as descriptor models [5]. A way to rewrite these models using the TS framework was proposed in [6]. Since the TS descriptor model keeps the nonlinear terms in the left-hand side, it can highly reduce the number of linear models in comparison with a classical TS model description [6]–[10]. In recent literature there are several works that concern discrete-time descriptor models [11]–[14], whereas other works refer to descriptor-redundancy, where the classical TS model is written in descriptor-like form to reduce the pessimism of the results [15]–[17].

One of the sources of conservativeness is the use of a quadratic Lyapunov function candidate [18]. For discrete-time TS models several improvements have been achieved with the use of non-quadratic Lyapunov functions. Generally, these Lyapunov functions share the same MFs as the TS model under consideration [19]–[23]. However, these advantages are not easily implemented in continuous-time TS models [18], [24]–[27]. More recently, polynomial approaches have been used [28]–[30], approaches exploiting the properties and shape of the MFs [31]–[34].

The aim of this work is to provide a systematic way to synthesize controllers for nonlinear discrete-time regular descriptor systems. This is motivated by mechanical systems where the left-hand side contains the mass matrix, which is regular and positive definitive [9], [35]. The basic idea is to rewrite the nonlinear descriptor model using TS descriptor representation. The controller design is based on two different non-quadratic Lyapunov functions, similar to those in [13], [36]; we also take advantage of the "delayed" approach presented in [21]; the main idea is to introduce past samples into the MFs of the Lyapunov function; thus bringing new controller structures including also information on past samples. This approach not only exploits the discrete-time nature of the problem; but from the point of view of the LMI constraint problem it brings relaxed results by increasing the number of decision variables while the number of LMI constraints remains the same. The new approaches are based on the well-known Finsler's Lemma, which avoids the explicit substitution of the closed-loop dynamics and allows adding slack variables [36]–[38]. In addition, this work provides a systematic analysis on how to choose the past samples involved in the MFs of the controller gains and the Lyapunov matrix, thus generalizing existing results in the literature [36].

The paper is organized as follows: Section II introduces the TS descriptor model, and provides some properties and the notation used along the work, together with a motivating example; Section III presents the main results for controller design for discrete-time TS descriptor models with illustrative examples; Section IV extends the results to H_∞ attenuation and robust control; Section V concludes the paper.

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II. Preliminaries

A. TS descriptor models

Consider the nonlinear descriptor model:

$$E\left(x(k)\right)x(k+1)=A\left(x(k)\right)x(k)+B\left(x(k)\right)u(k), \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state vector, $u(k)\in\mathbb{R}^m$ is the control input vector, and k is the current sample. Matrices $A(x)$, $B(x)$, and $E(x)$ are assumed to be smooth in a compact set Ω_x of the state space including the origin. Moreover, matrix $E(x)$ is assumed to be nonsingular for all $x(k)$ in the considered compact set Ω_x; this is motivated by mechanical systems, in which $E(x)$ contains the mass matrix and is therefore nonsingular [9], [31], see Example 1 hereafter.

Nonlinear terms are assumed to be bounded, i.e., $nl(\cdot)\in\left[\underline{nl},\overline{nl}\right]$. Using the sector nonlinearity approach [2] the p nonlinear terms in the right-hand side and the p_e nonlinear terms in the left-hand side are captured via the nonlinear membership functions (MFs) $h_i(\cdot)$, $i\in\{1,\ldots,2^p\}$ and $v_j(\cdot)$, $j\in\{1,\ldots,2^{p_e}\}$. These MFs hold the convex sum property, i.e., $\sum_{i=1}^{r}h_i(\cdot)=1$, $h_i(\cdot)\geq 0$, $\sum_{j=1}^{r_e}v_j(\cdot)=1$, $v_j(\cdot)\geq 0$ in the compact set Ω_x; the number of linear models in the right-hand side and in the left-hand side is $r=2^p$ and $r_e=2^{p_e}$, respectively. In addition, the MFs depend on the premise vector $z(k)$ which is assumed to be known.

Using the sector nonlinearity approach, an exact representation of the discrete-time nonlinear descriptor model (1) in the compact set Ω_x is [6]:

$$\sum_{j=1}^{r_e}v_j\left(z(k)\right)E_jx(k+1)=\sum_{i=1}^{r}h_i\left(z(k)\right)\left(A_ix(k)+B_iu(k)\right), \tag{2}$$

where matrices A_i, B_i, $i\in\{1,\ldots,r\}$ represent the *i*-th linear right-hand side model and E_j, $j\in\{1,\ldots,r_e\}$ represent the *j*-th linear left-hand side model of the TS descriptor model.

An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left-hand side. Arguments will be omitted when their meaning is evident.

B. Properties and lemmas

In order to obtain LMI conditions, MFs are usually dropped out from the expression; to this end the following sum relaxation scheme will be employed.

Lemma 1. [40] (Relaxation Lemma). Let $\Upsilon_{i_1i_2}$ be matrices of proper dimensions where $i_1,i_2\in\{1,\ldots,r\}$. If

$$\Upsilon_{i_1i_1}<0,\quad\forall i_1,\quad \frac{2}{r-1}\Upsilon_{i_1i_1}+\Upsilon_{i_1i_2}+\Upsilon_{i_2i_1}<0,\quad i_1\neq i_2, \tag{3}$$

then $\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)\Upsilon_{i_1i_2}<0$ holds.

In the next sections we will also use the following lemma and properties.

Lemma 2. [37] (Finsler's Lemma). Let $x\in\mathbb{R}^n$, $Q=Q^T\in\mathbb{R}^{n\times n}$, and $R\in\mathbb{R}^{m\times n}$ such that $rank(R)<n$; the following expressions are equivalent:

a) $x^TQx<0$, $\forall x\in\left\{x\in\mathbb{R}^n : x\neq 0, Rx=0\right\}$.

b) $\exists M\in\mathbb{R}^{n\times m}:Q+MR+R^TM^T<0$.

Property 1. Let $Q=Q^T>0$ and R be matrices of appropriate size. The following expression holds:
$(R-Q)^TQ^{-1}(R-Q)\geq 0\Leftrightarrow R^TQ^{-1}R\geq R+R^T-Q$.

Property 2. Let $Q=Q^T>0$, R and M be matrices of appropriate size. The following expression holds:
$R^TM+M^TR\leq R^TQR+M^TQ^{-1}M$.

C. Motivation

Of course, considering $E\left(x(k)\right)$ invertible for all trajectories such that $x\in\Omega_x$, (1) can be written in a classical TS form:

$$\begin{aligned}x(k+1)&=E^{-1}(x)A(x)x(k)+E^{-1}(x)B(x)u(k)\\&=\bar{A}(x)x(k)+\bar{B}(x)u(k),\end{aligned} \tag{4}$$

therefore the usual results apply directly on (4). Nevertheless, even if (1) and (4) are perfectly equivalent in Ω_x, the derivation of an equivalent TS model from the two nonlinear models can bring very different results and the two models directly influence the quality and the conservatism of the results. For example, the complexity of the LMI problems for control increases directly if the input matrix $B(x)$ is state dependent. This is due to the fact that cross products will occur [2], [6] and relaxations on double-sums must be used [40]. Therefore, if we consider (1) with a constant B its representation in (4) will be state dependent as $\bar{B}(x)=E^{-1}(x)B$. What we claim is that the closer the TS structure is to the nonlinear model, the best it is.

The following example issued from a real mechanical system shows the interest of the approach. We present the complexity of the TS models – classical vs. descriptor – and the feasibility results for various LMI constraints problems.

Example 1. Let us consider the continuous-time nonlinear descriptor system of the human stance presented in [9]. Using the Euler's approximation: $\dot{x}(t)=\left(x(k+1)-x(k)\right)/T_s$ a discrete-time descriptor system (1) is obtained:

$$E(x)x(k+1)=A(x)x(k)+Bu(k), \tag{5}$$

with $A(x)=\begin{bmatrix}I & T_s\times I\\ T_sG(x_1,x_2) & M(x_1,x_2)\end{bmatrix}$, $B=\begin{bmatrix}0\\ T_sR\end{bmatrix}$, and $E(x)=\begin{bmatrix}I & 0\\ 0 & M(x_1,x_2)\end{bmatrix}$; where $R=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$ is the linking in the joint torques, $M\left(x_1,x_2\right)=\begin{bmatrix} a & c\cos\left(x_1-x_2\right) \\ c\cos\left(x_1-x_2\right) & b \end{bmatrix}$ is the inertia matrix, the gravitational matrix is defined by $G\left(x_1,x_2\right)=\begin{bmatrix} d\sin\left(x_1\right)/x_1 & 0 \\ 0 & e\sin\left(x_2\right)/x_2 \end{bmatrix}$, and T_s is the sampling time. The Coriolis effect has been neglected, similar to [9]. The scalars $a=l_1+m_1K^2L_1^2+m_2L_1^2$, $b=l_2+m_2L_2^2$, $c=m_2L_1L_2$, $d=\left(m_2+m_1K\right)gL_1$, $e=m_2gL_2$ are defined in [9]: $K=0.525$, $L_1=0.87\ m$, $L_2=0.26\ m$, $I_1=1.25\ kg/m^2$, $I_2=2.32\ kg/m^2$, $m_1=21.87\ kg$, and $m_2=45.87\ kg$.

Using the sector nonlinearity approach, an exact representation (2) in $\Omega_x=\mathbb{R}^4$ gives $r_e=2$ due to the term $\cos\left(x_1-x_2\right)$ in $E\left(x\right)$; and $r=8$ due to nonlinearities $\cos\left(x_1-x_2\right)$, $\sin\left(x_1\right)/x_1$ and $\sin\left(x_2\right)/x_2$ in $A\left(x\right)$; notice also that the input matrix B is constant. As $E\left(x\right)$ and $A\left(x\right)$ share the nonlinearity $\cos\left(x_1-x_2\right)$, the TS descriptor representation of (5) has 8 linear descriptor models.

A "classical" form (4) can be obtained with: $E^{-1}\left(x\right)=\begin{bmatrix} I & 0 \\ 0 & M^{-1}\left(x_1,x_2\right) \end{bmatrix}$, $\Delta=ab-c^2\cos\left(x_1-x_2\right)^2$, and $M^{-1}\left(x_1,x_2\right)=\frac{1}{\Delta}\begin{bmatrix} b & -c\cos\left(x_1-x_2\right) \\ -c\cos\left(x_1-x_2\right) & a \end{bmatrix}$; therefore (4) writes: $\bar{A}\left(x\right)=\begin{bmatrix} I & T_s\times I \\ T_sM^{-1}\left(x_1,x_2\right)G\left(x_1,x_2\right) & I \end{bmatrix}$, and $\bar{B}\left(x\right)=\begin{bmatrix} 0 \\ T_sM^{-1}\left(x_1,x_2\right)R \end{bmatrix}$; thus nonlinearities come from:

$$M^{-1}\left(x_1,x_2\right)G\left(x_1,x_2\right)$$
$$=\frac{1}{\Delta}\begin{bmatrix} bd\frac{\sin\left(x_1\right)}{x_1} & -ce\cos\left(x_1-x_2\right)\frac{\sin\left(x_2\right)}{x_2} \\ -cd\cos\left(x_1-x_2\right)\frac{\sin\left(x_1\right)}{x_1} & ae\frac{\sin\left(x_2\right)}{x_2} \end{bmatrix}$$
and
$$M^{-1}\left(x_1,x_2\right)R=\frac{1}{\Delta}\begin{bmatrix} b & -b-c\cos\left(x_1-x_2\right) \\ -c\cos\left(x_1-x_2\right) & a \end{bmatrix}.$$

Applying the sector nonlinearity approach requires $r=2^4=16$ linear models. For example one can use the nonlinear terms $\sin\left(x_1\right)/x_1$, $\sin\left(x_2\right)/x_2$, $\cos\left(x_1-x_2\right)$, and $1/\Delta=1/\left(ab-c^2\cos\left(x_1-x_2\right)^2\right)$. Note that the number of linear models increased from 8 to 16 and that the input matrix now also contains nonlinear terms. These two facts will have a direct impact on the conservatism of the solutions.

Consider for the classical TS representation the conditions in [19]–[21], and [23], which are the main state of art conditions, together with Lemma 1. Using these conditions ***no solution*** is available; the complexity of the problem and the results are given in Table I. Considering the descriptor representation and without any relaxation scheme – as there is no cross-product due to the fact that B is constant – the following conditions can be used [13]:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j_1=1}^{r_e}h_{i_1}\left(z\left(k\right)\right)h_{i_2}\left(z\left(k+1\right)\right)v_{j_1}\left(z\left(k\right)\right)\times$$
$$\times\begin{bmatrix} -G_{j_1}-G_{j_1}^T+P_{i_1} & (*) \\ A_{i_1}G_{j_1}+BK_{i_1j_1} & -E_{j_1}P_{i_2}-P_{i_2}E_{j_1}^T+P_{i_2} \end{bmatrix}<0. \tag{6}$$

The number of LMIs is reduced and feasible solutions are obtained.

TABLE I
RESULTS FOR EXAMPLE 1, WHERE NQ STANDS FOR NON-QUADRATIC

Approach	Nr. of sums	Nr. of LMIs	Feasible solution
NQ Theorem 5 in [20] + Lemma 1	3	$r^3+r=4112$	No
NQ Theorem 1 in [19] + Lemma 1	4	$r^4+r^2=65792$	Numerical problems
NQ Theorem 1 in [23] + Lemma 1	4	$r^4+r^2=65792$	Numerical problems
NQ Theorem 1 in [21] + Lemma 1	3	$r^3+r=4112$	No
NQ [13] (TS Descriptor)	4	$r_er^2+r=136$	Yes

This example pointed out the motivation for deriving specific tools – controller, estimation, LMI conditions – for TS descriptor structure with $E\left(x\right)$ being regular. ◊

Let us recall the basic idea in [21], where the delayed MFs in the Lyapunov function were introduced for observer design for classical TS models. Two options for the Lyapunov functions are possible:

- OP1. $V\left(x\left(k\right)\right)=x^T\left(k\right)\left(\sum_{i=1}^{r}h\left(z\left(k\right)\right)P_i\right)^{-1}x\left(k\right)$ $=x^T\left(k\right)P_{z(k)}^{-1}x\left(k\right).$
- OP2. $V\left(x\left(k\right)\right)=x^T\left(k\right)\left(\sum_{i=1}^{r}h\left(z\left(k-1\right)\right)P_i\right)^{-1}x\left(k\right)$ $=x^T\left(k\right)P_{z(k-1)}^{-1}x\left(k\right).$

Case 1: the control law $u\left(k\right)=K_{z(k)}G_{z(k)}^{-1}x\left(k\right)$ together with the Lyapunov function OP1 yield condition: $\begin{bmatrix} -G_{z(k)}-G_{z(k)}^T+P_{z(k)} & (*) \\ A_{z(k)}G_{z(k)}+B_{z(k)}K_{z(k)} & -P_{z(k+1)} \end{bmatrix}<0$.

Case 2: considering a new control law $u\left(k\right)=K_{z(k)z(k-1)}G_{z(k)z(k-1)}^{-1}x\left(k\right)$ and the Lyapunov function OP2, the conditions are: $\begin{bmatrix} -G_{z(k)z(k-1)}-G_{z(k)z(k-1)}^T+P_{z(k-1)} & (*) \\ A_{z(k)}G_{z(k)z(k-1)}+B_{z(k)}K_{z(k)z(k-1)} & -P_{z(k)} \end{bmatrix}<0$.

These two conditions show the main interest of using delayed Lyapunov functions. In fact, the second one (Case 2) allows a double time description $z(k)z(k-1)$ in the control. Theoretically, it is also possible to introduce $u(k)=K_{z(k)z(k+1)}G_{z(k)z(k+1)}^{-1}x(k)$ in the first one (Case 1), but it clearly results in a non causal control. Note that the number of LMI conditions remains the same in both cases.

In Section 3, the advantages of delayed Lyapunov functions are extended to TS descriptor systems.

D. Notations

For sake of brevity and clarity, the following notations will be used throughout this paper [36].

Definition 1. (Multiple sum) A multiple sum with n_{Υ_h} terms and delays evaluated at sample k is of the form:

$$\Upsilon_{H_0^{\Upsilon}}=\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_{n_{\Upsilon_h}}=1}^{r}h_{i_1}\left(z(k+d_1)\right)\times$$
$$\times h_{i_2}\left(z(k+d_2)\right)\cdots h_{i_{n_{\Upsilon_h}}}\left(z\left(k+d_{n_{\Upsilon_h}}\right)\right)\Upsilon_{i_1 i_2\cdots i_{n_{\Upsilon_h}}},$$

where H_0^{Υ} is the multiset of delays $H_0^{\Upsilon}=\left\{d_1,d_2,\ldots,\mathrm{d}_{n_{\Upsilon_h}}\right\}$, $d_{(\cdot)}\in\square$. The definition of $\mathcal{V}_0^{\Upsilon}$ is similar, i.e., $\Upsilon_{\mathcal{V}_0^{\Upsilon}}=\sum_{j_1=1}^{r}\cdots\sum_{j_{n_{\Upsilon_h}}=1}^{r}v_{j_1}\left(z(k+d_1)\right)\cdots v_{j_{n_{\Upsilon_v}}}\left(z\left(k+d_{n_{\Upsilon_h}}\right)\right)\Upsilon_{j_1\cdots j_{n_{\Upsilon_v}}}$.

Definition 2. (Multiset of delays) H_0^{Υ} denotes the multiset containing the delays in the multiple sum involving Υ at sample k. H_α^{Υ} denotes the multiset containing the delays in the sum Υ at sample $k+\alpha$.

Definition 3. (Cardinality) The cardinality of a multiset H, $|H|=n_H$, is defined as the number of elements in H.

Definition 4. (Index set) The index set of a multiple sum Υ_H is $\mathbf{I}_H=\left\{i_j:i_j=1,2,\ldots,r,j=1,2,\ldots,|H|\right\}$, the set of all indices that appear in the sum. An element $\boldsymbol{i}$ is a multiindex.

Definition 5. (Multiplicity) The multiplicity of an element x in a multiset H, $\mathbf{1}_H(x)$ denotes the number of times this element appears in the multiset H.

Definition 6. (Union) The union of two multisets H_A and H_B is $H_C=H_A\cup H_B$, such that: $\forall x\in H_C:\mathbf{1}_{H_C}(x)=\max\left\{\mathbf{1}_{H_A}(x),\mathbf{1}_{H_B}(x)\right\}$.

Definition 7. (Intersection) The intersection of two multisets H_A and H_A is $H_C=H_A\cap H_B$ such that $\forall x\in H_C$: $\mathbf{1}_{H_C}(x)=\min\left\{\mathbf{1}_{H_A}(x),\mathbf{1}_{H_B}(x)\right\}$.

Definition 8. (Sum) The sum of two multisets H_A and H_B is $H_C=H_A\oplus H_B$ s. t. $\forall x\in H_C:\mathbf{1}_{H_C}(x)=\mathbf{1}_{H_A}(x)+\mathbf{1}_{H_B}(x)$.

The following example illustrates the previous definitions.

Example 2. Consider the multiple sum

$$\Upsilon_{H_0^{\Upsilon}}=\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{i_5=1}^{r}\sum_{i_6=1}^{r}h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)h_{i_3}\left(z(k-1)\right)\times$$
$$\times h_{i_4}\left(z(k-2)\right)h_{i_5}\left(z(k-3)\right)h_{i_6}\left(z(k-3)\right)\Upsilon_{i_1i_2i_3i_4i_5i_6}.$$

Then, H_0^{Υ} is given by $H_0^{\Upsilon}=\{0,0,-1,-2,-3,-3\}$, or $H_\alpha^{\Upsilon}=\{\alpha,\alpha,\alpha-1,\alpha-2,\alpha-3,\alpha-3\}$. The cardinality of H_0^{Υ} is $\left|H_0^{\Upsilon}\right|=n_{\Upsilon_h}=6$. The index set of the multiple sum $\Upsilon_{H_0^{\Upsilon}}$ is $\mathbf{I}_{H_0^{\Upsilon}}=\left\{i_j:i_j=1,2,\ldots,r,j=1,2,\ldots,6\right\}$. The multiplicity of the elements in H_0^{Υ} is $\mathbf{1}_{H_0^{\Upsilon}}(0)=2$, $\mathbf{1}_{H_0^{\Upsilon}}(-1)=1$, $\mathbf{1}_{H_0^{\Upsilon}}(-2)=1$, and $\mathbf{1}_{H_0^{\Upsilon}}(-3)=2$. Now, let H_A and H_B be two multisets defined as $H_A=\{0,0,-1,-2,-3\}$ and $H_B=\{0,-3,-4\}$. The union of these multisets is $H_A\cup H_B=\{0,0,-1,-2,-3,-4\}$, the intersection is $H_A\cap H_B=\{0,-3\}$, and their sum is $H_A\oplus H_B=\{0,0,0,-1,-2,-3,-3,-4\}$. ◊

Definition 9. (Projection of an index) The projection of the index $\boldsymbol{i}\in\mathbf{I}_{H_A}$ to the multiset of delays H_B, $pr_{H_B}^{\boldsymbol{i}}$, is the part of the index that corresponds to the delays in $H_A\cap H_B$.

Considering the definitions above, the discrete-time TS descriptor model (2) can be written as

$$E_{\mathcal{V}_0^E}x(k+1)=A_{H_0^A}x(k)+B_{H_0^B}u(k), \tag{7}$$

with $\mathcal{V}_0^E=H_0^A=H_0^B=\{0\}$, i.e., the system matrices are without delays.

The next section presents the main results. These are based on Finsler's Lemma and the use of two non-quadratic Lyapunov functions together with the so-called non-PDC.

III. Main results

A. Controller design via two Lyapunov functions

In what follows, for design purposes, consider the non-PDC control law:

$$u(k)=K_{H_0^K\mathcal{V}_0^K}G_{H_0^G\mathcal{V}_0^G}^{-1}x(k), \tag{8}$$

where $K_{H_0^K\mathcal{V}_0^K}$ and $G_{H_0^G\mathcal{V}_0^G}$ are matrices of appropriate dimensions to be determined. The regularity of $G_{H_0^G\mathcal{V}_0^G}$ will be discussed further on. Obviously, for causality these matrices cannot contain "positive delays" which correspond to future samples [21], [36]. The delays are given by the multisets H_0^K, H_0^G, $\mathcal{V}_0^K$, and $\mathcal{V}_0^G$. Thus, the closed-loop of the model (7) under the control law (8) is

$$E_{\mathcal{V}_0^E}x(k+1)=\left(A_{H_0^A}+B_{H_0^B}K_{H_0^K\mathcal{V}_0^K}G_{H_0^G\mathcal{V}_0^G}^{-1}\right)x(k). \tag{9}$$

Example 3. Recall that the multisets for the system matrices are $\mathcal{V}_0^E=H_0^A=H_0^B=\{0\}$ and by choosing $H_0^K=H_0^G=\mathcal{V}_0^K=\mathcal{V}_0^G=\{0\}$ for the controller gains, the closed-loop TS descriptor (9) renders:

$$\sum_{j_1=1}^{r_e}v_{j_1}\left(z(k)\right)E_{j_1}x(k+1)=\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j_1=1}^{r_e}h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)v_{j_1}\left(z(k)\right)$$
$$\times\left(A_{i_1}+B_{i_2}K_{i_2j_1}\times\left(\sum_{i_2=1}^{r}\sum_{j_1=1}^{r_e}h_{i_2}\left(z(k)\right)v_{j_1}\left(z(k)\right)G_{i_2j_1}\right)^{-1}\right)x(k)$$

which is the same as Case 1 of [13]. ◊

The analysis of the closed-loop model is done through the direct Lyapunov method. To this end, two different Lyapunov functions are proposed:

Case 1: $V\left(x(k)\right)=x^T(k)P^{-1}_{H_0^P\mathrm{V}_0^P}x(k)$,

Case 2: $V\left(x(k)\right)=x^T(k)G^{-T}_{H_0^G\mathrm{V}_0^G}P_{H_0^P\mathrm{V}_0^P}G^{-1}_{H_0^G\mathrm{V}_0^G}x(k)$,

where $P_{\boldsymbol{i}_0^P,\boldsymbol{j}_0^P}=P^T_{\boldsymbol{i}_0^P,\boldsymbol{j}_0^P}>0$, $\boldsymbol{i}\in\mathbf{I}_{H_0^P}$, $\boldsymbol{j}\in\mathbf{I}_{\mathrm{V}_0^P}$, and $G_{H_0^G\mathrm{V}_0^G}$ is the same matrix as in the controller (8).

Thereinafter, results are based on Lemma 2. It is important to remark that via Lemma 2 it is possible to avoid explicit substitution of the closed-loop dynamics and it facilitates dealing with the nonlinear matrix $E(x)$.

Considering Case 1, the following result can be stated:

Lemma 3. The closed-loop TS descriptor model (9) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P^T_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P=pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P=pr^{\boldsymbol{j}}_{\mathrm{V}_k^P}$, $k=0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K=pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K=pr^{\boldsymbol{j}}_{\mathrm{V}_0^K}$, and $G_{\boldsymbol{i}_0^G,\boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G=pr^{\boldsymbol{i}}_{H_0^G}$, $\boldsymbol{j}_0^G=pr^{\boldsymbol{j}}_{\mathrm{V}_0^G}$, $\boldsymbol{i}\in\mathbf{I}_{H_\Gamma}$, $\boldsymbol{j}\in\mathbf{I}_{\mathrm{V}_\Gamma}$, where $H_\Gamma=H_0^P\cup H_1^P\cup\left(H_0^B\oplus H_0^K\right)\cup\left(H_0^A\oplus H_0^G\right)$, $\mathrm{V}_\Gamma=\mathrm{V}_0^P\cup\mathrm{V}_1^P\cup\mathrm{V}_0^K\cup\mathrm{V}_0^G\cup\mathrm{V}_0^E$ such that

$$\begin{bmatrix}-G_{H_0^G\mathrm{V}_0^G}-G^T_{H_0^G\mathrm{V}_0^G}+P_{H_0^P\mathrm{V}_0^P} & (*)\\ A_{H_0^A}G_{H_0^G\mathrm{V}_0^G}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K} & -E_{\mathrm{V}_0^E}P_{H_1^P\mathrm{V}_1^P}-P_{H_1^P\mathrm{V}_1^P}E^T_{\mathrm{V}_0^E}+P_{H_1^P\mathrm{V}_1^P}\end{bmatrix}<0 \tag{10}$$

The number of sums involved in (10) for MFs $h(\cdot)$ and $v(\cdot)$ is $n_{H\mathrm{V}}=\left|H_\Gamma\right|+\left|\mathrm{V}_\Gamma\right|$.

Proof. The variation of the non-quadratic Lyapunov function $V\left(x(k)\right)=x^T(k)P^{-1}_{H_0^P\mathrm{V}_0^P}x(k)$ writes

$$\Delta V\left(x(k)\right)=\begin{bmatrix}x(k)\\x(k+1)\end{bmatrix}^T\begin{bmatrix}-P^{-1}_{H_0^P\mathrm{V}_0^P} & 0\\ 0 & P^{-1}_{H_1^P\mathrm{V}_1^P}\end{bmatrix}\begin{bmatrix}x(k)\\x(k+1)\end{bmatrix}<0. \tag{11}$$

Rewriting (9) as an equality constraint results in

$$\begin{bmatrix}A_{H_0^A}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K}G^{-1}_{H_0^G\mathrm{V}_0^G} & -E_{\mathrm{V}_0^E}\end{bmatrix}\begin{bmatrix}x(k)\\x(k+1)\end{bmatrix}=0. \tag{12}$$

Through Lemma 2, inequality (11) under constraint (12) is equivalent to:

$$\begin{bmatrix}-P^{-1}_{H_0^P\mathrm{V}_0^P} & 0\\ 0 & P^{-1}_{H_1^P\mathrm{V}_1^P}\end{bmatrix}+M_{(\cdot)}\begin{bmatrix}A_{H_0^A}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K}G^{-1}_{H_0^G\mathrm{V}_0^G} & -E_{\mathrm{V}_0^E}\end{bmatrix}+(*)<0, \tag{13}$$

with $M_{(\cdot)}\in\mathbb{R}^{2n\times n}$. Using the congruence property with matrix $diag\left(G^T_{H_0^G\mathrm{V}_0^G},P_{H_1^P\mathrm{V}_1^P}\right)$ yields

$$\begin{bmatrix}-G^T_{H_0^G\mathrm{V}_0^G}P^{-1}_{H_0^P\mathrm{V}_0^P}G_{H_0^G\mathrm{V}_0^G} & 0\\ 0 & P_{H_1^P\mathrm{V}_1^P}\end{bmatrix}+\begin{bmatrix}G^T_{H_0^G\mathrm{V}_0^G} & 0\\ 0 & P_{H_1^P\mathrm{V}_1^P}\end{bmatrix}M_{(\cdot)}\times \\ \times\begin{bmatrix}A_{H_0^A}G_{H_0^G\mathrm{V}_0^G}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K} & -E_{\mathrm{V}_0^E}P_{H_1^P\mathrm{V}_1^P}\end{bmatrix}+(*)<0. \tag{14}$$

The choice $M_{(\cdot)}=\begin{bmatrix}0\\P^{-1}_{H_1^P\mathrm{V}_1^P}\end{bmatrix}$ and applying Property 1 yields directly (10). Note also that if (10) is satisfied, its first block gives: $G_{H_0^G\mathrm{V}_0^G}+G^T_{H_0^G\mathrm{V}_0^G}>P_{H_0^P\mathrm{V}_0^P}>0$ and therefore $G_{H_0^G\mathrm{V}_0^G}$ is always regular thus concluding the proof. □

A refined result can be obtained by modifying the matrix $M_{(\cdot)}$ and the matrix used for congruence in (13) as follows:

Theorem 1. The closed-loop TS descriptor model (9) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P^T_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P=pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P=pr^{\boldsymbol{j}}_{\mathrm{V}_k^P}$, $k=0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K=pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K=pr^{\boldsymbol{j}}_{\mathrm{V}_0^K}$, $G_{\boldsymbol{i}_0^G,\boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G=pr^{\boldsymbol{i}}_{H_0^G}$, $\boldsymbol{j}_0^G=pr^{\boldsymbol{j}}_{\mathrm{V}_0^G}$, and $F_{\boldsymbol{i}_0^F,\boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F=pr^{\boldsymbol{i}}_{H_0^F}$, $\boldsymbol{j}_0^F=pr^{\boldsymbol{j}}_{\mathrm{V}_0^F}$, $\boldsymbol{i}\in\mathbf{I}_{H_\Gamma}$, $\boldsymbol{j}\in\mathbf{I}_{\mathrm{V}_\Gamma}$, where:

$H_\Gamma=H_0^P\cup H_1^P\cup\left(H_0^B\oplus H_0^K\right)\cup\left(H_0^A\oplus H_0^G\right)\cup H_0^F$,

$\mathrm{V}_\Gamma=\mathrm{V}_0^P\cup\mathrm{V}_1^P\cup\mathrm{V}_0^K\cup\mathrm{V}_0^G\cup\left(\mathrm{V}_0^E\oplus\mathrm{V}_0^F\right)$ such that

$$\begin{bmatrix}-G_{H_0^G\mathrm{V}_0^G}-G^T_{H_0^G\mathrm{V}_0^G}+P_{H_0^P\mathrm{V}_0^P} & (*) & (*)\\ A_{H_0^A}G_{H_0^G\mathrm{V}_0^G}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K} & -E_{\mathrm{V}_0^E}F_{H_0^F\mathrm{V}_0^F}+(*) & (*)\\ 0 & F_{H_0^F\mathrm{V}_0^F} & -P_{H_1^P\mathrm{V}_1^P}\end{bmatrix}<0 \tag{15}$$

The number of sums involved in (15) for MFs $h(\cdot)$ and $v(\cdot)$ is $n_{H\mathrm{V}}=\left|H_\Gamma\right|+\left|\mathrm{V}_\Gamma\right|$.

Proof. Consider (13), and $M_{(\cdot)}=\begin{bmatrix}0\\F^{-T}_{H_0^F\mathrm{V}_0^F}\end{bmatrix}$. Congruence with matrix $diag\left(G^T_{H_0^G\mathrm{V}_0^G},F^T_{H_0^F\mathrm{V}_0^F}\right)$ gives:

$$\begin{bmatrix}G^T_{H_0^G\mathrm{V}_0^G}P^{-1}_{H_0^P\mathrm{V}_0^P}G_{H_0^G\mathrm{V}_0^G} & (*)\\ A_{H_0^A}G_{H_0^G\mathrm{V}_0^G}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K} & \begin{pmatrix}-E_{\mathrm{V}_0^E}F_{H_0^F\mathrm{V}_0^F}-F^T_{H_0^F\mathrm{V}_0^F}E^T_{\mathrm{V}_0^E}\\+F^T_{H_0^F\mathrm{V}_0^F}P^{-1}_{H_1^P\mathrm{V}_1^P}F_{H_0^F\mathrm{V}_0^F}\end{pmatrix}\end{bmatrix}<0. \tag{16}$$

Applying Property 1 on the first block of (16) gives:

$$\begin{bmatrix}-G_{H_0^G\mathrm{V}_0^G}-G^T_{H_0^G\mathrm{V}_0^G}+P_{H_0^P\mathrm{V}_0^P} & (*)\\ A_{H_0^A}G_{H_0^G\mathrm{V}_0^G}+B_{H_0^B}K_{H_0^K\mathrm{V}_0^K} & \begin{pmatrix}-E_{\mathrm{V}_0^E}F_{H_0^F\mathrm{V}_0^F}-F^T_{H_0^F\mathrm{V}_0^F}E^T_{\mathrm{V}_0^E}\\+F^T_{H_0^F\mathrm{V}_0^F}P^{-1}_{H_1^P\mathrm{V}_1^P}F_{H_0^F\mathrm{V}_0^F}\end{pmatrix}\end{bmatrix}<0 \tag{17}$$

Finally, the Schur complement applied on (17) gives (15).□

Remark 1. The inclusion of the free matrix $F_{H_0^F\mathrm{V}_0^F}$ provides extra degrees of freedom, with respect to Lemma 3, while the number of LMI constraints remains the same. Also it can be seen that, under the same multisets, Theorem 1 always includes Lemma 3.

A similar procedure can be applied for the Lyapunov function in Case 2.

Theorem 2. The closed-loop TS descriptor model (9) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}=P^T_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P=pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P=pr^{\boldsymbol{j}}_{\mathrm{V}_k^P}$, $G_{\boldsymbol{i}_k^G,\boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G=pr^{\boldsymbol{i}}_{H_k^G}$, $\boldsymbol{j}_k^G=pr^{\boldsymbol{j}}_{\mathrm{V}_k^G}$, $k=0,1$, and

$K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K = pr_{\mathrm{V}_0^K}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbf{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K\right) \cup \left(H_0^A \oplus H_0^G\right) \cup H_1^G$, $\mathrm{V}_\Gamma = \mathrm{V}_0^P \cup \mathrm{V}_1^P \cup \mathrm{V}_0^K \cup \mathrm{V}_0^G \cup \left(\mathrm{V}_0^E \oplus \mathrm{V}_1^G\right)$ such that

$$\begin{bmatrix} -P_{H_0^P \mathrm{V}_0^P} & (*) \\ A_{H_0^A} G_{H_0^G \mathrm{V}_0^G} + B_{H_0^B} K_{H_0^K \mathrm{V}_0^K} & -E_{\mathrm{V}_0^E} G_{H_1^G \mathrm{V}_1^G} - G_{H_1^G \mathrm{V}_1^G}^T E_{\mathrm{V}_0^E}^T + P_{H_1^P \mathrm{V}_1^P} \end{bmatrix} < 0 \tag{18}$$

The number of sums involved in (18) for MFs $h(\cdot)$ and $v(\cdot)$ is $n_{H\mathrm{V}} = |H_\Gamma| + |\mathrm{V}_\Gamma|$.

Proof. The variation of the non-quadratic Lyapunov function $V(x(k)) = x^T(k) G_{H_0^G \mathrm{V}_0^G}^{-T} P_{H_0^P \mathrm{V}_0^P} G_{H_0^G \mathrm{V}_0^G}^{-1} x(k)$ gives:

$$X^T(\cdot) \begin{bmatrix} -G_{H_0^G \mathrm{V}_0^G}^{-T} P_{H_0^P \mathrm{V}_0^P} G_{H_0^G \mathrm{V}_0^G}^{-1} & 0 \\ 0 & G_{H_1^G \mathrm{V}_1^G}^{-T} P_{H_1^P \mathrm{V}_1^P} G_{H_1^G \mathrm{V}_1^G}^{-1} \end{bmatrix} X(\cdot) < 0, \tag{19}$$

with $X(\cdot) = \begin{bmatrix} x^T(k) & x^T(k+1) \end{bmatrix}^T$. Then, constraint (12) together with inequality (19) and using Lemma 2 gives

$$\begin{gathered} \begin{bmatrix} -G_{H_0^G \mathrm{V}_0^G}^{-T} P_{H_0^P \mathrm{V}_0^P} G_{H_0^G \mathrm{V}_0^G}^{-1} & 0 \\ 0 & G_{H_1^G \mathrm{V}_1^G}^{-T} P_{H_1^P \mathrm{V}_1^P} G_{H_1^G \mathrm{V}_1^G}^{-1} \end{bmatrix} \\ + M_{(\cdot)} \begin{bmatrix} A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathrm{V}_0^K} G_{H_0^G \mathrm{V}_0^G}^{-1} & -E_{\mathrm{V}_0^E} \end{bmatrix} + (*) < 0, \end{gathered} \tag{20}$$

with $M_{(\cdot)} \in \mathbb{R}^{2n \times n}$. Using the congruence property with matrix $diag\left(G_{H_0^G \mathrm{V}_0^G}^T, G_{H_1^G \mathrm{V}_1^G}^T\right)$ and choosing $M_{(\cdot)} = \begin{bmatrix} 0 \\ G_{H_1^G \mathrm{V}_1^G}^{-T} \end{bmatrix}$ gives directly (18), thus concluding the proof.$\square$

Remark 2. Note that the classical TS model is a special case of the TS descriptor one when $E_{\mathrm{V}_0^E} = I$, $\mathrm{V}_\Gamma = \varnothing$, where $\varnothing$ stands for the empty set; therefore Theorems 1 and 2 recover their respective theorems in [36].

An expansion of the Lemma 1 for two double sums follows.

Lemma 4. Consider a two double sum co-negativity problem where $i_1, i_2 \in \{1, \ldots, r\}$, $j_1, j_2 \in \{1, \ldots, r_e\}$.

$$\begin{aligned} \Upsilon_{hh}^{vv} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k)) \Upsilon_{i_1 i_2 j_1 j_2} < 0. \end{aligned} \tag{21}$$

Sufficient conditions for (21) to hold are:

$$\begin{aligned} & \Upsilon_{i_1 i_1 j_1 j_1} < 0, \quad \forall i_1, j_1, \\ & \frac{2}{r_e - 1} \Upsilon_{i_1 i_1 j_1 j_1} + \Upsilon_{i_1 i_1 j_1 j_2} + \Upsilon_{i_1 i_1 j_2 j_1} < 0, \quad \forall i_1, \quad j_1 \neq j_2, \\ & \frac{2}{r - 1} \Upsilon_{i_1 i_1 j_1 j_1} + \Upsilon_{i_1 i_2 j_1 j_1} + \Upsilon_{i_2 i_1 j_1 j_1} < 0, \quad \forall j_1, \quad i_1 \neq i_2, \\ & \frac{4}{(r_e - 1)(r - 1)} \Upsilon_{i_1 i_1 j_1 j_1} + \frac{2}{r_e - 1}\left(\Upsilon_{i_1 i_2 j_1 j_1} + \Upsilon_{i_2 i_1 j_1 j_1}\right) \\ & \quad + \frac{2}{r - 1}\left(\Upsilon_{i_1 i_1 j_1 j_2} + \Upsilon_{i_1 i_1 j_2 j_1}\right) + \Upsilon_{i_1 i_2 j_1 j_2} + \Upsilon_{i_2 i_1 j_1 j_2} \\ & \quad + \Upsilon_{i_1 i_2 j_2 j_1} + \Upsilon_{i_2 i_1 j_2 j_1} < 0, \quad i_1 \neq i_2, \quad j_1 \neq j_2, \end{aligned} \tag{22}$$

Proof. See Appendix A.$\square$

B. Discussion

At this point, let us clarify how to select the multisets involved in the control law and in the Lyapunov functions. To this end, an example is employed. Consider the closed-loop system (9) with $\mathrm{V}_0^E = H_0^A = H_0^B = \{0\}$ and the multisets $H_0^G = H_0^K = H_0^F = \{0, -1\}$, $\mathrm{V}_0^G = \mathrm{V}_0^F = \mathrm{V}_0^K = \{0, -1\}$, and $H_0^P = \mathrm{V}_0^P = \{-1\}$, i.e.,

$$P_{H_0^P \mathrm{V}_0^P} = P_{\{-1\},\{-1\}} = \sum_{i_1=1}^{r} \sum_{j_1=1}^{r_e} h_{i_1}(z(k-1)) v_{j_1}(z(k-1)) P_{i_1 j_1},$$

$$\begin{aligned} K_{H_0^K \mathrm{V}_0^K} = K_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k-1)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k-1)) K_{i_1 i_2 j_1 j_2}, \end{aligned}$$

$$\begin{aligned} G_{H_0^G \mathrm{V}_0^G} = G_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k-1)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k-1)) G_{i_1 i_2 j_1 j_2}, \end{aligned}$$

$$\begin{aligned} F_{H_0^G \mathrm{V}_0^G} = F_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k-1)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k-1)) F_{i_1 i_2 j_1 j_2}. \end{aligned}$$

Then, the conditions of Lemma 3 are

$$\begin{bmatrix} -G_{\{0,-1\},\{0,-1\}} + (*) + P_{\{-1\},\{-1\}} & (*) \\ A_{\{0\}} G_{\{0,-1\},\{0,-1\}} + B_{\{0\}} K_{\{0,-1\},\{0,-1\}} & -E_{\{0\}} P_{\{0\},\{0\}} + (*) + P_{\{0\},\{0\}} \end{bmatrix} < 0,$$

or

$$\begin{aligned} \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{i_3=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_3=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_3}(z(k-1)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_3}(z(k-1)) \times \\ & \times \begin{bmatrix} -G_{i_2 i_3 j_2 j_3} - G_{i_2 i_3 j_2 j_3}^T + P_{i_3, j_3} & (*) \\ A_{i_1} G_{i_2 i_3 j_2 j_3} + B_{i_1} K_{i_2 i_3 j_2 j_3} & -E_{j_1} P_{i_2 j_2} + (*) + P_{i_2 j_2} \end{bmatrix} < 0. \end{aligned}$$

The conditions of Theorem 1 are

$$\begin{bmatrix} \begin{pmatrix} -G_{\{0,-1\},\{0,-1\}} + (*) \\ + P_{\{-1\},\{-1\}} \end{pmatrix} & (*) & (*) \\ \begin{pmatrix} A_{\{0\}} G_{\{0,-1\},\{0,-1\}} \\ + B_{\{0\}} K_{\{0,-1\},\{0,-1\}} \end{pmatrix} & -E_{\{0\}} F_{\{0,-1\},\{0,-1\}} + (*) & (*) \\ 0 & F_{\{0,-1\},\{0,-1\}} & -P_{\{0\},\{0\}} \end{bmatrix} < 0,$$

or

$$\begin{aligned} \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{i_3=1}^{r} \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_3=1}^{r_e} & h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_3}(z(k-1)) \times \\ & \times v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_3}(z(k-1)) \times \\ & \times \begin{bmatrix} -G_{i_2 i_3 j_2 j_3} + (*) + P_{i_3 j_3} & (*) & (*) \\ A_{i_1} G_{i_2 i_3 j_2 j_3} + B_{i_1} K_{i_2 i_3 j_2 j_3} & -E_{j_1} F_{i_2 i_3 j_2 j_3} + (*) & (*) \\ 0 & F_{i_2 i_3 j_2 j_3} & -P_{i_2 j_2} \end{bmatrix} < 0. \end{aligned} \tag{23}$$

Similarly, the conditions of Theorem 2 are

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}\sum_{i_4=1}^{r}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_3=1}^{r_e}\sum_{j_4=1}^{r_e} h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)\times$$
$$\times h_{i_3}\left(z(k+1)\right)h_{i_4}\left(z(k-1)\right)v_{j_1}\left(z(k)\right)\times$$
$$\times v_{j_2}\left(z(k)\right)v_{j_3}\left(z(k+1)\right)v_{j_4}\left(z(k-1)\right)\times \tag{24}$$
$$\times\begin{bmatrix} -P_{i_3 j_3} & (*) \\ A_{i_1}G_{i_2 i_4 j_2 j_4}+B_{i_1}K_{i_2 i_3 j_2 j_3} & -E_{j_1}G_{i_2 i_3 j_2 j_3}+(*)+P_{i_2 j_2}\end{bmatrix}<0.$$

Recall that the maximum number of sums in Lemma 3 is: $n_{H\mathcal{V}}=\left|H_\Gamma\right|+\left|\mathcal{V}_\Gamma\right|\le 2n_{P_h}+2n_{P_v}+n_{K_h}+n_{K_v}+n_{G_h}+n_{G_v}+2$, which for Theorem 1 becomes: $n_{H\mathcal{V}}\le 2n_{P_h}+2n_{P_v}+n_{K_h}+n_{K_v}+n_{G_h}+n_{G_v}+n_{F_h}+n_{F_v}+2$, while for Theorem 2 it is: $n_{H\mathcal{V}}\le 2n_{P_h}+2n_{P_v}+n_{K_h}+n_{K_v}+2n_{G_h}+2n_{G_v}+2$; this of course, considering that $\mathcal{V}_0^E=H_0^A=H_0^B=\{0\}$, i.e., the considered system is (2), without delays in the system matrices. The relations above show that the maximum number of sums in Theorem 1 is smaller than the maximum number of sums in Theorem 2 (for instance, see (23) and (24)).

Therefore multisets in $K_{H_0^K\mathcal{V}_0^K}$, $G_{H_0^G\mathcal{V}_0^G}$, $F_{H_0^F\mathcal{V}_0^F}$, and $P_{H_0^P\mathcal{V}_0^P}$ should be chosen such that sum relaxations can be used and considering the number of sums and the computational complexity of the resulting LMI, with the goal to profit from the co-negativity problem as in Lemma 4.

What are the "good" multisets? Without considering solver limitations, the following reasoning applies:

Step 1: since $\mathcal{V}_0^E=H_0^A=H_0^B=\{0\}$, multisets H_0^K, H_0^G, H_0^F, $\mathcal{V}_0^K$, $\mathcal{V}_0^G$, and $\mathcal{V}_0^F$ should contain $\{0\}$. Double sum relaxations and the maximum number of variables should be used, but without increasing the number of sums. To illustrate these, consider $H_0^K=H_0^G=H_0^F=\mathcal{V}_0^K=\mathcal{V}_0^G=\mathcal{V}_0^F=\{0\}$. The conditions of Theorem 1 are

$$\begin{bmatrix} -G_{\{0\},\{0\}}-G_{\{0\},\{0\}}^T+P_{H_0^P\mathcal{V}_0^P} & (*) & (*) \\ A_{\{0\}}G_{\{0\},\{0\}}+B_{\{0\}}K_{\{0\},\{0\}} & -E_{\{0\}}F_{\{0\},\{0\}}+(*) & (*) \\ 0 & F_{\{0\},\{0\}} & -P_{H_1^P\mathcal{V}_1^P}\end{bmatrix}<0 \tag{25}$$

By selecting $H_0^P=\mathcal{V}_0^P=\{-1\}$, (25) gives

$$\begin{bmatrix} -G_{\{0\},\{0\}}-G_{\{0\},\{0\}}^T+P_{\{-1\},\{-1\}} & (*) & (*) \\ A_{\{0\}}G_{\{0\},\{0\}}+B_{\{0\}}K_{\{0\},\{0\}} & -E_{\{0\}}F_{\{0\},\{0\}}+(*) & (*) \\ 0 & F_{\{0\},\{0\}} & -P_{\{0\},\{0\}}\end{bmatrix}<0. \tag{26}$$

From (26) one can see that the co-negativity problem is in the form of (21). Expression (26) ends in six sums, i.e., for MFs $h(\cdot)$: $\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_3=1}^{r}h_{i_1}\left(z(k)\right)h_{i_2}\left(z(k)\right)h_{i_3}\left(z(k-1)\right)$ and $v(\cdot)$: $\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_3=1}^{r_e}v_{j_1}\left(z(k)\right)v_{j_2}\left(z(k)\right)v_{j_3}\left(z(k-1)\right)$.

Step 2: due to the structure (26), it is possible to add the delay $\{-1\}$ into each multiple sum $K_{H_0^K\mathcal{V}_0^K}$, $G_{H_0^G\mathcal{V}_0^G}$, $F_{H_0^F\mathcal{V}_0^F}$ without increasing the number of sums. Therefore we obtain:

$$\begin{bmatrix} \begin{pmatrix} -G_{\{0,-1\},\{0,-1\}}+(*) \\ +P_{\{-1\},\{-1\}}\end{pmatrix} & (*) & (*) \\ \begin{pmatrix} A_{\{0\}}G_{\{0,-1\},\{0,-1\}} \\ +B_{\{0\}}K_{\{0,-1\},\{0,-1\}}\end{pmatrix} & -E_{\{0\}}F_{\{0,-1\},\{0,-1\}}+(*) & (*) \\ 0 & F_{\{0,-1\},\{0,-1\}} & -P_{\{0\},\{0\}}\end{bmatrix}<0. \tag{27}$$

Step 3: since the multiple sum $F_{H_0^F\mathcal{V}_0^F}$ does not multiply $A_{H_0^A}$ and $B_{H_0^B}$, one can add $\{0\}$ in H_0^F; similarly for the multiple sums $K_{H_0^K\mathcal{V}_0^K}$ and $G_{H_0^G\mathcal{V}_0^G}$: one can add $\{0\}$ in $\mathcal{V}_0^K$ and $\mathcal{V}_0^G$, respectively. Thus the "good" multisets for this problem are:

$$\begin{bmatrix} \begin{pmatrix} -G_{\{0,-1\},\{0,0,-1\}}+(*) \\ +P_{\{-1\},\{-1\}}\end{pmatrix} & (*) & (*) \\ \begin{pmatrix} A_{\{0\}}G_{\{0,-1\},\{0,0,-1\}} \\ +B_{\{0\}}K_{\{0,-1\},\{0,0,-1\}}\end{pmatrix} & -E_{\{0\}}F_{\{0,0,-1\},\{0,-1\}}+(*) & (*) \\ 0 & F_{\{0,0,-1\},\{0,-1\}} & -P_{\{0\},\{0\}}\end{bmatrix}<0.$$

Table II shows how the number of decision variables changes at each step.

TABLE II
NUMBER OF DECISION VARIABLES IN THEOREM 1.

Step	Nr. of LMIs	Feasible solution
Step 1	$\left(0.5n\times(n+1)\right)\times(r\times r_e)$ $+2\times\left(n^2\right)\times(r\times r_e)$ $+(m\times n)\times(r\times r_e)$	3 sums in $h(\cdot)$ 3 sums in $v(\cdot)$
Step 2	$\left(0.5n\times(n+1)\right)\times(r\times r_e)$ $+2\times\left(n^2\right)\times\left(r^2\times r_e^2\right)$ $+(m\times n)\times\left(r^2\times r_e^2\right)$	3 sums in $h(\cdot)$ 3 sums in $v(\cdot)$
Step 3	$\left(0.5n\times(n+1)\right)\times(r\times r_e)$ $+\left(n^2\right)\times\left(r^2\times r_e^3\right)$ $+\left(n^2\right)\times\left(r^3\times r_e^2\right)$ $+(m\times n)\times\left(r^2\times r_e^3\right)$	3 sums in $h(\cdot)$ 3 sums in $v(\cdot)$

Let us analyze the options for Theorem 2 starting with multisets $H_0^K=H_0^G=\mathcal{V}_0^K=\mathcal{V}_0^G=\{0\}$, which give:

$$\begin{bmatrix} -P_{H_0^P\mathcal{V}_0^P} & (*) \\ A_{\{0\}}G_{\{0\},\{0\}}+B_{\{0\}}K_{\{0\},\{0\}} & -E_{\{0\}}G_{\{1\},\{1\}}+(*)+P_{H_1^P\mathcal{V}_1^P}\end{bmatrix}<0. \tag{28}$$

Since there is no double sum in $v(\cdot)$ at the current sample k, it is possible to add $\{0\}$ in $\mathcal{V}_0^K$, i.e., $\mathcal{V}_0^K=\left(\mathcal{V}_0^E\oplus\mathcal{V}_0^G\right)$. Then (28) gives:

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0,0\}} & -E_{\{0\}} G_{\{1\},\{1\}} + (*) + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \quad (29)$$

Expression (29) ends in three sums for $h(\cdot)$, i.e., $\sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \sum_{i_3=1}^{r} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_3}(z(k+1))$ and three for $v(\cdot)$: $\sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_3=1}^{r_e} v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_3}(z(k+1))$, 6 sums in total, the same as for Theorem 1. To keep the same number of sums, the best solution for the Lyapunov multiple sum is $H_0^P = \mathcal{V}_0^P = \{0\}$. Finally, (29) renders:

$$\begin{bmatrix} -P_{\{0\},\{0\}} & (*) \\ A_{\{0\}} G_{\{0\},\{0\}} + B_{\{0\}} K_{\{0\},\{0,0\}} & -E_{\{0\}} G_{\{1\},\{1\}} + (*) + P_{\{1\},\{1\}} \end{bmatrix} < 0. \quad (30)$$

Note that positive delays cannot be added to K and G, since the information of the future states is not available. Table III summarizes these results for an arbitrary cardinality of the multisets.

TABLE III
HOW TO SELECT MULTISETS FOR THEOREM 1 AND THEOREM 2.

Multiple sum	Multisets in Theorem 1	Multisets in Theorem 2
$P_{H_0^P \mathcal{V}_0^P}$	$H_0^P = \{-1,-1,\dots,-1\}$ $\lvert H_0^P \rvert = n_{P_h}$	$H_0^P = \{0,0,\dots,0\}$ $\lvert H_0^P \rvert = n_{P_h}$
	$\mathcal{V}_0^P = \{-1,-1,\dots,-1\}$ $\lvert \mathcal{V}_0^P \rvert = n_{P_v}$	$\mathcal{V}_0^P = \{0,0,\dots,0\}$ $\lvert \mathcal{V}_0^P \rvert = n_{P_v}$
$K_{H_0^K \mathcal{V}_0^K}$	$H_0^K = \{\underbrace{0,0,\dots,0}_{n_{P_h}}, \underbrace{-1,-1,\dots,-1}_{n_{P_h}}\}$ $\lvert H_0^K \rvert = 2n_{P_h}$	$H_0^K = \{\underbrace{0,0,\dots,0,}_{n_{P_h}}\}$ $\lvert H_0^K \rvert = n_{P_h}$
	$\mathcal{V}_0^K = \{0,\underbrace{0,0,\dots,0}_{n_{P_v}}, \underbrace{-1,-1,\dots,-1}_{n_{P_v}}\}$ $\lvert \mathcal{V}_0^K \rvert = 1+2n_{P_v}$	$\mathcal{V}_0^K = \{0,\underbrace{0,0,\dots,0}_{n_{P_v}}\}$ $\lvert \mathcal{V}_0^K \rvert = 1+n_{P_v}$
$G_{H_0^G \mathcal{V}_0^G}$	$H_0^G = \{\underbrace{0,0,\dots,0}_{n_{P_h}}, \underbrace{-1,-1,\dots,-1}_{n_{P_h}}\}$ $\lvert H_0^G \rvert = 2n_{P_h}$	$H_0^G = \{\underbrace{0,0,\dots,0,}_{n_{P_h}}\}$ $\lvert H_0^G \rvert = n_{P_h}$
	$\mathcal{V}_0^G = \{0,\underbrace{0,0,\dots,0}_{n_{P_v}}, \underbrace{-1,-1,\dots,-1}_{n_{P_v}}\}$..	$\mathcal{V}_0^G = \{\underbrace{0,0,\dots,0,}_{n_{P_v}}\}$ $\lvert \mathcal{V}_0^G \rvert = n_{P_v}$
$F_{H_0^F \mathcal{V}_0^F}$	$H_0^F = \{0,\underbrace{0,0,\dots,0}_{n_{P_h}}, \underbrace{-1,-1,\dots,-1}_{n_{P_h}}\}$ $\lvert H_0^F \rvert = 1+2n_{P_h}$	----------------
	$\mathcal{V}_0^F = \{\underbrace{0,0,\dots,0}_{n_{P_v}}, \underbrace{-1,-1,\dots,-1}_{n_{P_v}}\}$ $\lvert \mathcal{V}_0^F \rvert = 2n_{P_v}$	----------------

C. Examples

The following examples illustrate the performances of Lemma 3, Theorem 1, and Theorem 2.

Example 4. Consider a nonlinear descriptor system

$$E(x)x(k+1) = A(x)x(k) + B(x)u(k), \quad (31)$$

with matrices as follows:

$$E(x) = \begin{bmatrix} 1.4 + b\left(2/\left(1+x_1^2\right)-1\right) & 0.2 \\ -0.1 & 1.6 - a\left(2/\left(1+x_1^2\right)-1\right) \end{bmatrix},$$

$A(x) = \begin{bmatrix} -1+ax_2 & 0.3 \\ 1.4+bx_2 & 1.5 \end{bmatrix}$, and $B(x) = \begin{bmatrix} 0 \\ -1-0.3ax_2 \end{bmatrix}$; where a and b are real-valued parameters, $a,b \in [-1.5, 1.5]$. To obtain a TS descriptor model of the form (2), consider the MFs defined as $h_1 = (x_2+1)/2$, $h_2 = 1-h_1$, $v_1 = 1/\left(1+x_1^2\right)$, and $v_2 = 1-v_1$. These MFs hold the convex-sum property in the compact $\Omega_x = \{x : x_1 \in \mathbb{R}, \lvert x_2 \rvert \le 1\}$. Thus, (31) is equivalent to:

$$\sum_{j=1}^{2} v_j(z(k)) E_j x(k+1) = \sum_{i=1}^{2} h_i(z(k)) \left(A_i x(k) + B_i u(k)\right),$$

with $E_1 = \begin{bmatrix} 1.4+b & 0.2 \\ -0.1 & 1.6-a \end{bmatrix}$, $E_2 = \begin{bmatrix} 1.4-b & 0.2 \\ -0.1 & 1.6+a \end{bmatrix}$, $A_1 = \begin{bmatrix} -1+a & 0.3 \\ 1.4+b & 1.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1-a & 0.3 \\ 1.4-b & 1.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ -1-0.3a \end{bmatrix}$, and $B_2 = \begin{bmatrix} 0 \\ -1+0.3a \end{bmatrix}$. This example compares Lemma 3 and Theorem 1; for a "fair" comparison between them, multisets are chosen as follows:

- Ch1: Lemma 3 with $H_0^P = H_0^K = H_0^G = \{0\}$, $\mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$, and $\mathcal{V}_0^P = \varnothing$, which corresponds to Lemma 1 in [13]. The results are illustrated in Figure 1A.
- Ch2: Lemma 3 with $H_0^P = \mathcal{V}_0^P = \{-1\}$, $H_0^K = H_0^G = \{0,-1\}$, and $\mathcal{V}_0^K = \mathcal{V}_0^G = \{0,0,-1\}$. The results are illustrated in Figure 1A.

- Ch3: Theorem 1 with $H_0^P = H_0^K = H_0^G = \{0\}$, $\mathcal{V}_0^K = \mathcal{V}_0^G = \mathcal{V}_0^F = \{0\}$, $H_0^F = \{0,0,1\}$, and $\mathcal{V}_0^P = \mathcal{V}_0^F = \varnothing$, which corresponds to Theorem 1 in [13]. The results are shown in Figure 1B.
- Ch4: Theorem 1 with $H_0^P = \mathcal{V}_0^P = \{-1\}$, $H_0^K = H_0^G = \{0,-1\}$, $H_0^F = \{0,0,-1\}$, $\mathcal{V}_0^F = \{0,-1\}$, and $\mathcal{V}_0^K = \mathcal{V}_0^G = \{0,0,-1\}$. The results are shown in Figure 1B.

Choices Ch2 and Ch4 give 3 sums in $h(\cdot)$ and 3 sums in $v(\cdot)$. Figure 1 illustrates Remark 1. The feasible solutions for conditions of Lemma 3 are plotted in Figure 1A: Ch1 (∇) and Ch2 ($+$). Figure 1B shows the solution set for conditions via Theorem 1: Ch3 (□) and Ch4 ($\times$). Under the same number of sums, the solution set of Theorem 1 always includes the one of Lemma 3 (Remark 1).

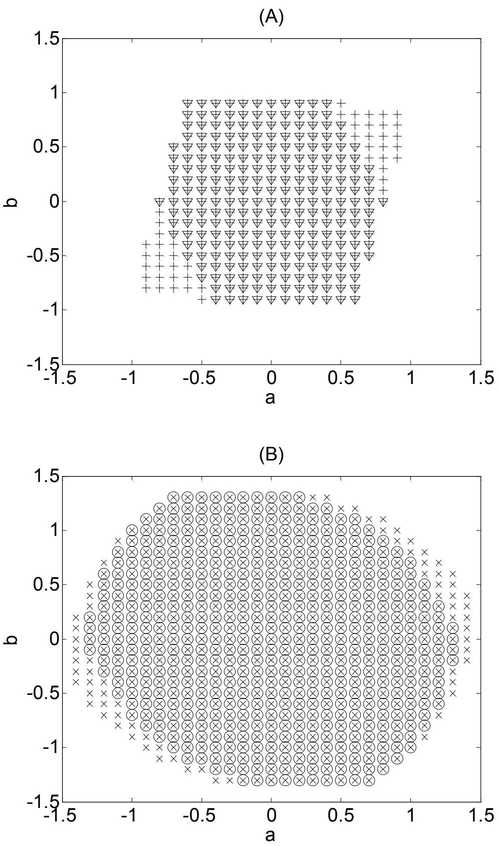

Figure 1. Feasible sets of parameter values in Example 4.

For parameter values $a = -1.3$ and $b = -0.5$ no solution was found using the conditions in Ch1, Ch2, and Ch3; while conditions of Ch4 provide a solution; some of the controller gains are: $K_{11111} = \begin{bmatrix} 0.57 & 0.88 \end{bmatrix}$, $K_{11211} = \begin{bmatrix} 0.77 & 1.55 \end{bmatrix}$, $K_{11212} = \begin{bmatrix} 0.43 & 1.36 \end{bmatrix}$, $K_{21212} = \begin{bmatrix} 0.09 & 0.63 \end{bmatrix}$, $G_{11111} = \begin{bmatrix} 0.02 & 0.04 \\ 0.09 & 0.32 \end{bmatrix}$, and $G_{12112} = \begin{bmatrix} 0.02 & 0.02 \\ 0.03 & 0.24 \end{bmatrix}$.

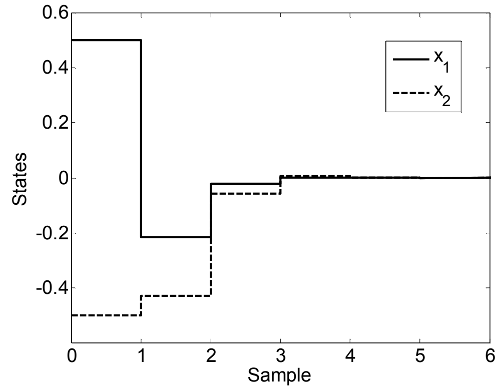

Figure 2. States simulation in Example 4.

Figure 2 shows the convergence of the states for initial conditions $x(0) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^T$; recall that results presented in [13] cannot provide a controller for the system when $a = -1.3$ and $b = -0.5$. ◊

Example 5. Consider (2) with $r = r_e = 2$, $A_1 = \begin{bmatrix} 1.18 - 0.2\beta & -1.31 \\ -0.33 & 0.23 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.69 & 1.41 \\ -1.17 & 1.43 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ -1.05 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 - 0.1\beta \\ 0 \end{bmatrix}$, $E_1 = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.36 \end{bmatrix}$, and $E_2 = \begin{bmatrix} 0.95 & 0 \\ 0 & 1 \end{bmatrix}$; where $\beta > 0$ is a real-valued parameter. Applying Theorem 2 with multisets:

- $H_0^P = H_0^K = H_0^G = \mathcal{V}_0^K = \{0\}$, $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$ (four sums are involved), the maximum value of β for which conditions were found feasible is $\beta = 0.86$. (Using the same number of sums the conditions of Theorem 1 are not feasible for any β).
- $H_0^P = H_0^G = H_0^K = \{0,-1\}$, $\mathcal{V}_0^K = \{0\}$, $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$ (five sums are involved), the maximum value of β for which conditions were found feasible is $\beta = 0.90$.
- $H_0^P = H_0^K = H_0^G = \mathcal{V}_0^P = \mathcal{V}_0^G = \{0\}$ and $\mathcal{V}_0^K = \{0,0\}$, both 3 sums in $h(\cdot)$ and $v(\cdot)$, the maximum value was $\beta = 1.86$.

IV. Extensions

An advantage of the TS-LMI framework is that different specifications can be implemented. This section shows the extensions of the previous results to H_∞ attenuation and robust control.

A. H_∞ attenuation

An external disturbance signal $w(k) \in \mathbb{R}^q$ implies the TS descriptor form:

$$\begin{aligned} E_{\mathcal{V}_0^E} x(k+1) &= A_{H_0^A} x(k) + B_{H_0^B} u(k) + D_{H_0^D} w(k) \\ y(k) &= C_{H_0^C} x(k) + J_{H_0^J} w(k), \end{aligned} \tag{32}$$

where $y(k) \in \mathbb{R}^o$ is the output vector, the multiple sums $C_{H_0^C}$,

$D_{H_0^D}$, and $J_{H_0^J}$ share definitions similar to $A_{H_0^A}$. Using the control law (8) for the TS descriptor (32) gives

$$\begin{aligned} E_{\mathrm{V}_0^E} x(k+1) &= \left(A_{H_0^A} + B_{H_0^B} K_{H_0^K \mathrm{V}_0^K} G^{-1}_{H_0^G \mathrm{V}_0^G} \right) x(k) + D_{H_0^D} w(k) \\ y(k) &= C_{H_0^C} x(k) + J_{H_0^J} w(k). \end{aligned} \tag{33}$$

The disturbance attenuation corresponds to the well-known condition:

$$\Delta V(x(k)) + y^T(k) y(k) - \gamma^2 w^T(k) w(k) < 0. \tag{34}$$

For Case 1 the following result can be stated:

Theorem 3. The closed-loop system (33) is asymptotically stable and the attenuation is at least γ if there exist $\gamma > 0$, $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P^T_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P = pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P = pr^{\boldsymbol{j}}_{\mathrm{V}_k^P}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K = pr^{\boldsymbol{j}}_{\mathrm{V}_0^K}$, $G_{\boldsymbol{i}_0^G, \boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr^{\boldsymbol{i}}_{H_0^G}$, $\boldsymbol{j}_0^G = pr^{\boldsymbol{j}}_{\mathrm{V}_0^G}$, and $F_{\boldsymbol{i}_0^F, \boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{H_0^F}$, $\boldsymbol{j}_0^F = pr^{\boldsymbol{j}}_{\mathrm{V}_0^F}$, $\boldsymbol{i} \in \mathbf{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K \right) \cup \left(H_0^G \oplus \left(H_0^A \cup H_0^C \right) \right) \cup H_0^F$, $\mathrm{V}_\Gamma = \mathrm{V}_0^P \cup \mathrm{V}_1^P \cup \mathrm{V}_0^K \cup \mathrm{V}_0^G \cup \left(\mathrm{V}_0^E \oplus \mathrm{V}_0^F \right)$ such that

$$\begin{bmatrix} -G_{H_0^G \mathrm{V}_0^G} - G^T_{H_0^G \mathrm{V}_0^G} + P_{H_0^P \mathrm{V}_0^P} & (*) & (*) & (*) & (*) \\ A_{H_0^A} G_{H_0^G \mathrm{V}_0^G} + B_{H_0^B} K_{H_0^K \mathrm{V}_0^K} & \Upsilon^{(2,2)} & (*) & (*) & (*) \\ 0 & \mathrm{F}_{H_0^F \mathrm{V}_0^F} & -P_{H_1^P \mathrm{V}_1^P} & (*) & (*) \\ 0 & D^T_{H_0^D} & 0 & -\gamma^2 I & (*) \\ C_{H_0^C} G_{H_0^G \mathrm{V}_0^G} & 0 & 0 & J_{H_0^J} & -I \end{bmatrix} < 0 \tag{35}$$

with $\Upsilon^{(2,2)} = -E_{\mathrm{V}_0^E} \mathrm{F}_{H_0^F \mathrm{V}_0^F} - F^T_{H_0^F \mathrm{V}_0^F} E^T_{\mathrm{V}_0^E}$.

Proof. The proof follows similar lines as Theorem 1.□

For Case 2 the following result can be established:

Theorem 4. The closed-loop system (33) is asymptotically stable and the attenuation is at least γ if there exist $\gamma > 0$, $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P^T_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P}$, $\boldsymbol{i}_k^P = pr^{\boldsymbol{i}}_{H_k^P}$, $\boldsymbol{j}_k^P = pr^{\boldsymbol{j}}_{\mathrm{V}_k^P}$, $G_{\boldsymbol{i}_k^G, \boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G = pr^{\boldsymbol{i}}_{H_k^G}$, $\boldsymbol{j}_k^G = pr^{\boldsymbol{j}}_{\mathrm{V}_k^G}$, $k = 0,1$, and $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr^{\boldsymbol{i}}_{H_0^K}$, $\boldsymbol{j}_0^K = pr^{\boldsymbol{j}}_{\mathrm{V}_0^K}$, $\boldsymbol{i} \in \mathbf{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup \left(H_0^B \oplus H_0^K \right) \cup \left(H_0^G \oplus \left(H_0^A \cup H_0^C \right) \right) \cup H_1^G$, $\mathrm{V}_\Gamma = \mathrm{V}_0^P \cup \mathrm{V}_1^P \cup \mathrm{V}_0^K \cup \mathrm{V}_0^G \cup \left(\mathrm{V}_0^E \oplus \mathrm{V}_1^G \right)$ such that

$$\begin{bmatrix} -P_{H_0^P \mathrm{V}_0^P} & (*) & (*) & (*) \\ \Upsilon^{(2,1)} & -E_{\mathrm{V}_0^E} G_{H_1^G \mathrm{V}_1^G} + (*) + P_{H_1^P \mathrm{V}_1^P} & (*) & (*) \\ 0 & D^T_{H_0^D} & -\gamma^2 I & (*) \\ C_{H_0^C} G_{H_0^G \mathrm{V}_0^G} & 0 & J_{H_0^J} & -I \end{bmatrix} < 0, \tag{36}$$

with $\Upsilon^{(2,1)} = A_{H_0^A} G_{H_0^G \mathrm{V}_0^G} + B_{H_0^B} K_{H_0^K \mathrm{V}_0^K}$.

Proof. The proof follows similar lines as Theorem 2.□

Remark 3. For comparable given multisets the results obtained through conditions of Theorem 1 and Theorem 2 are not equivalent [36]. The same reasoning applies for the case of H_∞ attenuation. One theorem may succeed while the other one fails (see Example 7).

The following examples illustrate the performances of Theorem 3 and Theorem 4. Let us recall the motivating Example 1. The local matrices for the TS descriptor representation are given for $T_s = 0.3s$.

Example 6. Consider the discrete-time nonlinear descriptor model of the human stance from Example 1. Using the sector nonlinearity approach in $\Omega_x = \mathbb{R}^4$, the following local matrices were obtained: in the left-hand side $r_e = 2$, $E_1 = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} a & c \\ c & b \end{bmatrix} \end{bmatrix}$, and $E_2 = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} a & -c \\ -c & b \end{bmatrix} \end{bmatrix}$; in the right-hand side $r = 8$ with matrices $A_1 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix} & \begin{bmatrix} a & c \\ c & b \end{bmatrix} \end{bmatrix}$,

$$A_2 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} d & 0 \\ 0 & \varphi e \end{bmatrix} & \begin{bmatrix} a & c \\ c & b \end{bmatrix} \end{bmatrix}, \quad A_3 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} \varphi d & 0 \\ 0 & e \end{bmatrix} & \begin{bmatrix} a & c \\ c & b \end{bmatrix} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} \varphi d & 0 \\ 0 & \varphi e \end{bmatrix} & \begin{bmatrix} a & c \\ c & b \end{bmatrix} \end{bmatrix}, A_5 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} \varphi d & 0 \\ 0 & e \end{bmatrix} & \begin{bmatrix} a & -c \\ -c & b \end{bmatrix} \end{bmatrix},$$

$$A_6 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} d & 0 \\ 0 & \varphi e \end{bmatrix} & \begin{bmatrix} a & -c \\ -c & b \end{bmatrix} \end{bmatrix}, A_7 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} \varphi d & 0 \\ 0 & e \end{bmatrix} & \begin{bmatrix} a & -c \\ -c & b \end{bmatrix} \end{bmatrix}$$

$$A_8 = \begin{bmatrix} I & T_s \\ T_s \begin{bmatrix} \varphi d & 0 \\ 0 & \varphi e \end{bmatrix} & \begin{bmatrix} a & -c \\ -c & b \end{bmatrix} \end{bmatrix}, \quad B = T_s \begin{bmatrix} 0 \\ R \end{bmatrix}, \quad J = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix},$$

$D_1 = D_2 = D_3 = D_4 = T_s \times \begin{bmatrix} 0.7 & 1-\alpha & 0.5 & 0.9 \end{bmatrix}^T$,

$D_5 = D_6 = D_7 = D_8 = T_s \times \begin{bmatrix} 0.7 & 1+\alpha & 0.5 & 0.9 \end{bmatrix}^T$,

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, where α is a real-valued parameter. The value $\varphi = -0.2172$ is the lower bound on the nonlinear terms $\sin(x_1)/x_1$ and $\sin(x_2)/x_2$. The scalars are defined in Example 1. Considering that matrices B and C are constant –there is no cross-product–, different configurations for multisets have been tested:

- Conf1: Theorem 3 with $H_0^P = H_0^K = \mathrm{V}_0^K = \mathrm{V}_0^G = \{0\}$, $H_0^F = \{0,1\}$, and $H_0^G = \mathrm{V}_0^P = \mathrm{V}_0^F = \varnothing$. There are no double-sums involved at the current sample time.
- Conf2: Theorem 3 with $H_0^G = H_0^P = \mathrm{V}_0^P = \{-1\}$, $H_0^K = H_0^F = \mathrm{V}_0^F = \{0,-1\}$, and $\mathrm{V}_0^K = \mathrm{V}_0^G = \{0,0,-1\}$ which gives a problem with 2 sums in $h(\cdot)$ and 3 sums in $v(\cdot)$. Note that Lemma 1 is applicable over the double sum in $v(\cdot)$ at the current sample time.

- Conf3: Theorem 4 with multisets as $H_0^P = H_0^K = \mathcal{V}_0^K = \{0\}$ and $H_0^G = \mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$. There are no double-sums involved at the current sample time.
- Conf4: Theorem 4 with multisets $H_0^P = H_0^K = \mathcal{V}_0^P = \mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$ and $H_0^G = \varnothing$. There are no double-sums involved at the current sample time.

Table 4 shows results for several parameter values when different multiset options have been tested:

TABLE IV
γ VALUES IN EXAMPLE 6.

α value	Conf1	Conf2	Conf3	Conf4
$\alpha = 0$	$\gamma = 0.9525$	$\gamma = 0.8325$	$\gamma = 0.9525$	$\gamma = 0.9523$
$\alpha = 0.3$	$\gamma = 1.0791$	$\gamma = 0.9546$	$\gamma = 1.0828$	$\gamma = 1.0827$
$\alpha = 0.7$	$\gamma = 1.2558$	$\gamma = 1.1261$	$\gamma = 1.2637$	$\gamma = 1.2636$
$\alpha = 1$	$\gamma = 1.3926$	$\gamma = 1.2592$	$\gamma = 1.4032$	$\gamma = 1.4032$

Note that Conf1 and Conf3 have the same co-negativity problem: $\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{j_1=1}^{r} h_{i_1}(z(k)) h_{i_2}(z(k+1)) v_{j_1}(z(k))$. However, for this example Conf1 provides better results than Conf3. Moreover, using the classical TS representation it is not possible to design a controller, see Example 1. ◊

The following numerical example illustrates the performances of Theorem 3 and Theorem 4. Different options for multisets have been tested:

- Opt1: Theorem 3 with $H_0^P = H_0^K = H_0^G = \{0\}$, $\mathcal{V}_0^K = \mathcal{V}_0^G = \{0\}$, $H_0^F = \{0,0,1\}$, and $\mathcal{V}_0^P = \mathcal{V}_0^F = \varnothing$ which gives a problem with 3 sums in $h(\cdot)$ and 1 sum in $v(\cdot)$.
- Opt2: Theorem 3 with $H_0^K = H_0^G = \mathcal{V}_0^F = \{0,-1\}$, $H_0^P = \mathcal{V}_0^P = \{-1\}$, and $\mathcal{V}_0^K = \mathcal{V}_0^G = H_0^F = \{0,0,-1\}$ which gives a problem with 3 sums in $h(\cdot)$ and 3 sums in $v(\cdot)$.
- Opt3: Theorem 4 with multisets as $H_0^P = H_0^G = H_0^K = \mathcal{V}_0^K = \{0\}$ and $\mathcal{V}_0^P = \mathcal{V}_0^G = \varnothing$ which gives a problem with 3 sums in $h(\cdot)$ and 1 sum in $v(\cdot)$.
- Opt4: Theorem 4 with multisets $H_0^P = H_0^K = H_0^G = \{0\}$, $\mathcal{V}_0^P = \mathcal{V}_0^G = \{0\}$, and $\mathcal{V}_0^K = \{0,0\}$ which gives a problem with 3 sums in $h(\cdot)$ and 3 sums in $v(\cdot)$.

Note that multisets for Opt1 and Opt3 correspond to those in Theorem 1 and 2 in [13].

Example 7. Consider the TS descriptor model (32) with $r = r_e = 2$, $E_1 = \begin{bmatrix} 0.9 & 0.1+\alpha \\ -0.4 & 1.1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.9 & 1.1 \\ -0.4 & 1.1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & -1.5 \\ 0 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & -1.5 \\ 2 & 0.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 1.28 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 & 0.43 \end{bmatrix}$, $D_1 = \begin{bmatrix} 0.23 & 0 \end{bmatrix}^T$, $D_2 = \begin{bmatrix} 0 & 0.12 \end{bmatrix}^T$, $J_1 = 0.12$, and $J_2 = 0.09+\alpha$, ; where α is a real-valued parameter. Table V shows the results for several parameter values when using the options above.

TABLE V
γ VALUES IN EXAMPLE 7.

α value	Opt1	Opt2	Opt3	Opt4
$\alpha = -1.5$	$\gamma = 2.73$	$\gamma = 1.79$	$\gamma = 2.44$	$\gamma = 2.18$
$\alpha = -1$	$\gamma = 1.23$	$\gamma = 1.12$	$\gamma = 1.27$	$\gamma = 1.21$
$\alpha = -0.5$	$\gamma = 0.69$	$\gamma = 0.64$	$\gamma = 0.76$	$\gamma = 0.72$
$\alpha = 0$	$\gamma = 0.53$	$\gamma = 0.50$	$\gamma = 0.56$	$\gamma = 0.56$
$\alpha = 0.5$	$\gamma = 0.77$	$\gamma = 0.77$	$\gamma = 0.77$	$\gamma = 0.77$

The obtained results illustrate Remark 3, for instance, for $\alpha = -1.5$, Opt3 has provided better γ attenuation than Opt1; while for $\alpha = -0.5$ Opt1 has given better result than Opt3. ◊

B. Robust control

Consider a TS descriptor model with uncertainties:

$$\left(E_{\mathcal{V}_0^E} + \Delta E\right)x(k+1) = \left(A_{H_0^A} + \Delta A\right)x(k) + \left(B_{H_0^B} + \Delta B\right)u(k), \quad (37)$$

with the uncertainties defined as $\Delta E = D_{\mathcal{V}_{0,e}^D}\Delta_e L_{\mathcal{V}_{0,e}^L}$, $\Delta A = D_{H_{0,a}^D}\Delta_a L_{H_{0,a}^L}$, and $\Delta B = D_{H_{0,b}^D}\Delta_b L_{H_{0,b}^L}$, with $\Delta_e^T\Delta_e < I$, $\Delta_a^T\Delta_a < I$, and $\Delta_b^T\Delta_b < I$. The uncertain model (37) under the control law (8) gives

$$\begin{aligned} &\left(E_{\mathcal{V}_0^E} + \Delta E\right)x(k+1) \\ &= \left(A_{H_0^A} + B_{H_0^B}K_{H_0^K\mathcal{V}_0^K}G_{H_0^G\mathcal{V}_0^G}^{-1} + \Delta A + \Delta B K_{H_0^K\mathcal{V}_0^K}G_{H_0^G\mathcal{V}_0^G}^{-1}\right)x(k). \end{aligned} \quad (38)$$

Then, for Case 1, the following result can be stated:

Theorem 5. The closed-loop system (38) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P} = P_{\boldsymbol{i}_k^P,\boldsymbol{j}_k^P}^T$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K,\boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K = pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_0^G,\boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G = pr_{\mathcal{V}_0^G}^{\boldsymbol{j}}$, $F_{\boldsymbol{i}_0^F,\boldsymbol{j}_0^F}$, $\boldsymbol{i}_0^F = pr_{H_0^F}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^F = pr_{\mathcal{V}_0^F}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,a}^\tau,\boldsymbol{j}_{0,a}^\tau}$, $\boldsymbol{i}_0^{\tau_a} = pr_{H_{0,a}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_a} = pr_{\mathcal{V}_{0,a}^\tau}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,b}^\tau,\boldsymbol{j}_{0,b}^\tau}$, $\boldsymbol{i}_0^{\tau_b} = pr_{H_{0,b}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_b} = pr_{\mathcal{V}_{0,b}^\tau}^{\boldsymbol{j}}$, and $\tau_{\boldsymbol{i}_{0.e}^\tau,\boldsymbol{j}_{0,e}^\tau}$, $\boldsymbol{i}_0^{\tau_e} = pr_{H_{0,e}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_e} = pr_{\mathcal{V}_{0,e}^\tau}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbf{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathcal{V}_\Gamma}$, where

$$\begin{aligned} H_\Gamma &= H_0^P \cup H_1^P \cup \left(H_0^K \oplus \left(H_0^B \cup H_{0,b}^L\right)\right) \cup \left(H_0^G \oplus \left(H_0^A \cup H_{0,\mathrm{a}}^L\right)\right) \\ &\cup H_0^F \cup \left(H_{0,a}^\tau \oplus H_{0,a}^D\right) \cup \left(H_{0,\mathrm{b}}^\tau \oplus H_{0,\mathrm{b}}^D\right) \cup H_{0,\mathrm{e}}^\tau, \end{aligned}$$

$$\begin{aligned} \mathcal{V}_\Gamma &= \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \left(\mathcal{V}_0^F \oplus \left(\mathcal{V}_0^E \cup \mathcal{V}_{0,e}^L\right)\right) \cup \left(\mathcal{V}_{0,\mathrm{e}}^\tau \oplus \mathcal{V}_{0,\mathrm{e}}^D\right) \\ &\cup \mathcal{V}_{0,\mathrm{a}}^\tau \cup \mathcal{V}_{0,\mathrm{b}}^\tau \end{aligned}$$

such that

$$\left[\begin{array}{cc|c|c|c} \Upsilon^{(1,1)} & (*) & & & \\ \Upsilon^{(2,1)} & -E_{\mathcal{V}_0^E}F_{H_0^F\mathcal{V}_0^F} - F_{H_0^F\mathcal{V}_0^F}^T E_{\mathcal{V}_0^E}^T & (*) & (*) & (*) \\ \hline \multicolumn{2}{c|}{\tilde{\mathrm{D}}} & -\tilde{\mathrm{T}} & (*) & (*) \\ \hline \multicolumn{2}{c|}{\tilde{\mathrm{G}}} & 0 & -\tilde{\mathrm{T}} & (*) \\ \hline \multicolumn{2}{c|}{\tilde{\mathrm{F}}} & 0 & 0 & -P_{H_1^P\mathcal{V}_1^P} \end{array}\right] < 0, \quad (39)$$

where

$$\Upsilon^{(1,1)} = -G_{H_0^G \mathcal{V}_0^G} - G_{H_0^G \mathcal{V}_0^G}^T + P_{H_0^P \mathcal{V}_0^P},\ \Upsilon^{(2,1)} = A_{H_0^A} G_{H_0^G \mathcal{V}_0^G} + B_{H_0^B} K_{H_0^K \mathcal{V}_0^K},$$

$$\tilde{\mathrm{D}} = \begin{bmatrix} 0 & \tau_{H_{0,a}^\tau \mathcal{V}_{0,a}^\tau} D_{H_{0,\mathrm{a}}^D}^T \\ 0 & \tau_{H_{0,\mathrm{b}}^\tau \mathcal{V}_{0,\mathrm{b}}^\tau} D_{H_{0,\mathrm{b}}^D}^T \\ 0 & \tau_{H_{0,\mathrm{e}}^\tau \mathcal{V}_{0,\mathrm{e}}^\tau} D_{\mathcal{V}_{0,\mathrm{e}}^D}^T \end{bmatrix}, \qquad \tilde{\mathrm{G}} = \begin{bmatrix} L_{H_{0,\mathrm{a}}^L} G_{H_0^G \mathcal{V}_0^G} & 0 \\ L_{H_{0,\mathrm{b}}^L} K_{H_0^K \mathcal{V}_0^K} & 0 \\ 0 & -L_{\mathcal{V}_{0,\mathrm{e}}^L} F_{H_0^F \mathcal{V}_0^F} \end{bmatrix},$$

$$\tilde{\mathrm{F}} = \begin{bmatrix} 0 & F_{H_0^F \mathcal{V}_0^F} \end{bmatrix}, \text{ and } \tilde{\mathrm{T}} = \begin{bmatrix} \tau_{H_{0,a}^\tau \mathcal{V}_{0,a}^\tau} & 0 & 0 \\ 0 & \tau_{H_{0,\mathrm{b}}^\tau \mathcal{V}_{0,\mathrm{b}}^\tau} & 0 \\ 0 & 0 & \tau_{H_{0,\mathrm{e}}^\tau \mathcal{V}_{0,\mathrm{e}}^\tau} \end{bmatrix}.$$

Proof. The proof follows similar lines as Theorem 1.□

For Case 2:

Theorem 6. The closed-loop system (38) is asymptotically stable if there exist $P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P} = P_{\boldsymbol{i}_k^P, \boldsymbol{j}_k^P}^T$, $\boldsymbol{i}_k^P = pr_{H_k^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^P = pr_{\mathcal{V}_k^P}^{\boldsymbol{j}}$, $G_{\boldsymbol{i}_k^G, \boldsymbol{j}_k^G}$, $\boldsymbol{i}_k^G = pr_{H_k^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_k^G = pr_{\mathcal{V}_k^G}^{\boldsymbol{j}}$, $k = 0,1$, $K_{\boldsymbol{i}_0^K, \boldsymbol{j}_0^K}$, $\boldsymbol{i}_0^K = pr_{H_0^K}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^K = pr_{\mathcal{V}_0^K}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,a}^\tau, \boldsymbol{j}_{0,a}^\tau}$, $\boldsymbol{i}_0^{\tau_a} = pr_{H_{0,a}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_a} = pr_{\mathcal{V}_{0,a}^\tau}^{\boldsymbol{j}}$, $\tau_{\boldsymbol{i}_{0,b}^\tau, \boldsymbol{j}_{0,b}^\tau}$, $\boldsymbol{i}_0^{\tau_b} = pr_{H_{0,b}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_b} = pr_{\mathcal{V}_{0,b}^\tau}^{\boldsymbol{j}}$, and $\tau_{\boldsymbol{i}_{0.e}^\tau, \boldsymbol{j}_{0,e}^\tau}$, $\boldsymbol{i}_0^{\tau_e} = pr_{H_{0,e}^\tau}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^{\tau_e} = pr_{\mathcal{V}_{0,e}^\tau}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbf{I}_{H_\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathcal{V}_\Gamma}$, where $H_\Gamma = H_0^P \cup H_1^P \cup H_1^G \cup \left(H_0^K \oplus \left(H_0^B \cup H_{0,b}^L\right)\right) \cup \left(H_0^G \oplus \left(H_0^A \cup H_{0,\mathrm{a}}^L\right)\right) \cup \left(H_{0,a}^\tau \oplus H_{0,a}^D\right) \cup \left(H_{0,\mathrm{b}}^\tau \oplus H_{0,\mathrm{b}}^D\right) \cup H_{0,\mathrm{e}}^\tau$, $\mathcal{V}_\Gamma = \mathcal{V}_0^P \cup \mathcal{V}_1^P \cup \mathcal{V}_0^K \cup \mathcal{V}_0^G \cup \mathcal{V}_{0,\mathrm{a}}^\tau \cup \mathcal{V}_{0,\mathrm{b}}^\tau \cup \left(\mathcal{V}_1^G \oplus \left(\mathcal{V}_0^E \cup \mathcal{V}_{0,e}^L\right)\right) \cup \left(\mathcal{V}_{0,\mathrm{e}}^\tau \oplus \mathcal{V}_{0,\mathrm{e}}^D\right)$ such that

$$\left[\begin{array}{cc|c|c} -P_{H_0^P \mathcal{V}_0^P} & (*) & (*) & (*) \\ \Upsilon^{(2,1)} & -E_{\mathcal{V}_0^E} G_{H_1^G \mathcal{V}_1^G} + (*) + P_{H_1^P \mathcal{V}_1^P} & & \\ \hline \multicolumn{2}{c|}{\tilde{\mathrm{D}}} & -\tilde{\mathrm{T}} & (*) \\ \hline \multicolumn{2}{c|}{\tilde{\mathrm{G}}} & 0 & -\tilde{\mathrm{T}} \end{array}\right] < 0, \tag{40}$$

where $\Upsilon^{(2,1)}$, $\tilde{\mathrm{D}}$, and $\tilde{\mathrm{T}}$ were defined in Theorem 5, and

$$\tilde{\mathrm{G}} = \begin{bmatrix} L_{H_{0,\mathrm{a}}^L} G_{H_0^G \mathcal{V}_0^G} & 0 \\ L_{H_{0,\mathrm{b}}^L} K_{H_0^K \mathcal{V}_0^K} & 0 \\ 0 & -L_{\mathcal{V}_{0,\mathrm{e}}^L} G_{H_1^G \mathcal{V}_1^G} \end{bmatrix}.$$

Proof. The proof follows similar lines as Theorem 2.□

V. CONCLUSIONS

A novel controller design for nonlinear descriptor systems in TS form has been presented. The benefits of the TS descriptor model instead of the classical TS model are pointed out via a biomechanical example. The proposed approaches exploit the discrete-time nature of the treated problem by adding delays in the MFs, thus relaxing previous conditions. Design conditions are given in terms of LMIs which can be solved via convex optimization techniques. The validity of the approaches is illustrated on a biomechanical example as well as on numerical examples.

APPENDIX A

Proof of Lemma 4: Applying Lemma 1 on the double convex sum of $h(z(k))$ in (21) yields

$$\Upsilon_{i_1 i_1}^{vv} < 0, \quad \forall i_1, \quad \frac{2}{r-1}\Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0, \quad i_1 \neq i_2. \tag{41}$$

Now taking the first inequality in (41) using Lemma 1 over the double convex sum of $v(z(k))$ it renders

$$\Upsilon_{i_1 i_1}^{vv} < 0 \rightarrow \begin{cases} \Upsilon_{i_1 i_1 j_1 j_1} < 0, \quad \forall i_1, j_1, \\ \dfrac{2}{r_e - 1}\Upsilon_{i_1 i_1 j_1 j_1} + \Upsilon_{i_1 i_1 j_1 j_2} + \Upsilon_{i_1 i_1 j_2 j_1} < 0, \quad \forall i_1, \quad j_1 \neq j_2. \end{cases}$$

Finally, following a similar procedure with the second inequality in (41), it ends in:

$$\frac{2}{r-1}\Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0$$

$$\Rightarrow \begin{cases} \dfrac{2}{r-1}\Upsilon_{i_1 i_1 j_1 j_1} + \Upsilon_{i_1 i_2 j_1 j_1} + \Upsilon_{i_2 i_1 j_1 j_1} < 0, \quad \forall j_1, \quad i_1 \neq i_2, \\ \dfrac{2}{r_e - 1}\left(\dfrac{2}{r-1}\Upsilon_{i_1 i_1 j_1 j_1} + \Upsilon_{i_1 i_2 j_1 j_1} + \Upsilon_{i_2 i_1 j_1 j_1}\right) \\ \quad + \left(\dfrac{2}{r-1}\Upsilon_{i_1 i_1 j_1 j_2} + \Upsilon_{i_1 i_2 j_1 j_2} + \Upsilon_{i_2 i_1 j_1 j_2}\right) \\ \quad + \left(\dfrac{2}{r-1}\Upsilon_{i_1 i_1 j_2 j_1} + \Upsilon_{i_1 i_2 j_2 j_1} + \Upsilon_{i_2 i_1 j_2 j_1}\right) < 0, \quad i_1 \neq i_2, \quad j_1 \neq j_2, \end{cases}$$

thus concluding the proof. □

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