Sliding mode observer based fuzzy control for TS systems

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***Abstract*—This paper presents a sliding mode observer based fuzzy control. The sliding mode observer is developed for a linear dominant system, but taking into account the model mismatch. After that a fuzzy state feedback controller is designed. To ensure the stability of the closed loop system in the presence of uncertainties, Lyapunov synthesis is used. The results are illustrated on a numerical example. Simulations on the nonlinear system are presented to demonstrate the effectiveness of the observer based control.**

***Index Terms*—sliding mode observer, state feedback control, linear dominant model**

I. Introduction

The interest in control and estimation of nonlinear models has increased considerably in recent years, in particular because of the use of robotic systems. Various control methods can be found in the literature as backstepping [1], PID control [2], fuzzy control [3], sliding mode controller [4], predictive control [5], etc.

For models that are highly nonlinear, analysis and control can be a difficult challenge. Uncertainties present in the model can affect the control of the system. Sliding mode observers are robust estimators that can obtain a good performance. Such observers can estimate the states despite the modeling uncertainties and external disturbances. In [6] a sliding mode controller with a sliding mode observer is presented that is robust with the respect to uncertainty. An extension to an affine model of this combination is studied in [7]. The work [8] compares a Luenberger observer with a sliding mode observer, and both are used in combination with a sliding mode controller. Extending the research, generally sliding mode observers are used with sliding mode controllers [9]–[12]. In this paper we propose to simplify the equations by using a linear dominant model for the sliding mode observer in combination with a fuzzy state feedback controller. For the observer we use as a reference the work presented in [13].

The goal is to develop a stabilizing fuzzy state feedback controller that uses the states estimated by a robust sliding mode observer. This considers the modeling errors and uncertainties that can appear, and minimizes their effect on the estimation error. We develop a fuzzy state feedback controller that uses the estimated states given by the robust observer to control the system.

For estimation purposes the nonlinear system is approximated with a linear dominant model that includes known modeling errors and also uncertainties. Then, the system is transformed such that the measured states are separated from the unmeasured ones. After that, the sliding mode observer is designed using Lyapunov synthesis following the development in [13]. Next, we consider a fuzzy state feedback controller that uses the states estimated by the robust observer. To prove the stability of the closed-loop system, a Lyapunov function is used.

The paper is structured as follows. In Section II the preliminaries are presented, together with the original sliding mode observer and the problem statement. Section III describes the main result: the improvement of the observer and the development of the feedback controller that uses the estimated states from the improved sliding observer. Section IV illustrates the result on a numerical nonlinear system. In Section V the conclusions and future work are presented.

Notations. We use standard notations, same as in [14]. Consider a real symmetric matrix $F = F^T \in R^{n\times n}$; $F > 0$ or $F < 0$ denotes that F is positive or negative definite, respectively. We denote with I the identity matrix, and with 0 the zero matrix of appropriate dimensions. $(*)$ denotes the symmetric term.

II. Preliminaries and problem statement

We consider a nonlinear system having the following form:

$$\begin{aligned}\dot{x} &= f(x,u) + R\xi;\\ y &= Cx\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input, f represents the nonlinear system and the uncertainties present in the model are denoted by ξ, which are considered to be magnitude bounded as $\|\xi\| \leq k_1\|y\|$, $k_1 \geq 0$. $y \in \mathbb{R}^{n_y}$ is the measured output vector and C is the output matrix. The nonlinear model (1) is approximated by the following fuzzy model:

$$\begin{aligned}\dot{x} =& \sum_{i=1}^{s} h_i(z)(A_i x + B_i u) + \psi + R\xi\\ y =& Cx\end{aligned} \tag{2}$$

where (A_i, B_i, C) are local models. R is the uncertainty distribution matrix, z is the vector of premise variables assumed to be known, the number of rules is denoted by s and h_i, $i = 1,..,s$, are nonlinear membership functions with the property $h_i(z) \in [0,1], i = 1,...,s, \sum_{i=1}^{s} h_i(z) = 1$. The error between the nonlinear model and fuzzy model is denoted with ψ and expressed as:

$$\psi = f(x,u) - \sum_{i=1}^{s} h_i(z)(A_i x + B_i u) \tag{3}$$

To simplify the notations, in what follows, all the convex sums present in the equations are denoted by the matrix name present in the sum and subscripts that denote the dependence on the current state. For instance, $F_z = \sum_{i=1}^{s} h_i(z(t))F_i$.

Using such notations, (2) can be written as:

$$\begin{aligned} \dot{x}(t) =& A_z x(t) + B_z u(t) + \psi + R\xi \\ y(t) =& Cx(t) \end{aligned} \tag{4}$$

As presented by [15], beside the fuzzy model, we also consider a linearly dominant model. Thus, we consider:

$$\dot{x} = A_j x + B_j u + \delta_j + \psi + R\xi \tag{5}$$

a model valid in a neighbourhood of 0. The error between the fuzzy model and the linear model is given by:

$$\delta_j = (A_z - A_j)x + (B_z - B_j)u \tag{6}$$

and is completely known.

Given the system (4), our goal is to design a sliding mode observer

$$\begin{aligned} \dot{\hat{x}} =& A_z \hat{x} + B_z u + \psi + G_1(y - \hat{y}) - G_2 N \\ y =& C\hat{x} \end{aligned} \tag{7}$$

with the gains G_1 for the linear linear and G_2 for the nonlinear part and N being a vector to deal with the uncertainties. We also consider a fuzzy state feedback controller that is based on the estimated states:

$$u = -K_z \hat{x} \tag{8}$$

that should asymptotically stabilize the closed-loop system.

Results are developed using the following lemma and properties:

Lemma 1. (Congruence) Having the matrix $P = P^T$ and a full column rank matrix Q, it holds that:

$$P > 0 \implies QPQ^T > 0$$

Property 1. The following property holds for any $Q = Q^T > 0$, A and B matrices of appropriate sizes:

$$A^T B + B^T A \leq A^T Q A + B^T Q^{-1} B$$

Property 2. The following property holds for any A and $B = B^T > 0$ matrices of appropriate sizes:

$$-A^T B^{-1} A \leq -A^T - A + B$$

In the following we describe briefly the sliding mode observer proposed by [13] that our developments are based on. Assuming there exists a linear transformation that will introduce the outputs as new system states:

$$v(t) = Tx(t) \tag{9}$$

with the condition that

$$CT^{-1} = \begin{bmatrix} 0 & I \end{bmatrix};\ TR\xi = \begin{bmatrix} 0 \\ \xi^{21} \end{bmatrix} \tag{10}$$

then, (5) can be rewritten as:

$$\dot{v} = TA_j T^{-1} v + TB_j u + T\delta_j + T\psi + TR\xi \tag{11}$$

where:

$$\begin{aligned} TA_j T^{-1} = \begin{bmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{bmatrix};\ TB_j = \begin{bmatrix} B_j^{11} \\ B_j^{21} \end{bmatrix} \\ T\delta_j = \begin{bmatrix} \delta_j^{11} \\ \delta_j^{21} \end{bmatrix};\ T\psi = \begin{bmatrix} \psi^{11} \\ \psi^{21} \end{bmatrix} \end{aligned} \tag{12}$$

Similarly,

$$TA_z T^{-1} = \begin{bmatrix} A_z^{11} & A_z^{12} \\ A_z^{21} & A_z^{22} \end{bmatrix};\ TB_z = \begin{bmatrix} B_z^{11} \\ B_z^{21} \end{bmatrix} \tag{13}$$

The model mismatch is:

$$\begin{aligned} \delta_j^{11} =& (A_z^{11} - A_j^{11})v^{11} + (A_z^{12} - A_j^{12})v^{21} \\ &+ (B_z^{11} - B_j^{11})u \\ \delta_j^{21} =& (A_z^{21} - A_j^{21})v^{11} + (A_z^{22} - A_j^{22})v^{21} \\ &+ (B_z^{21} - B_j^{21})u \end{aligned} \tag{14}$$

Using the notations (12) and (14), (11) can be expressed as:

$$\begin{aligned} \dot{v}^{11} =& A_j^{11} v^{11} + A_j^{12} v^{21} + B_j^{11} u + \delta_j^{11} + \psi^{11} \\ \dot{v}^{21} =& A_j^{21} v^{11} + A_j^{22} v^{21} + B_j^{21} u + \delta_j^{21} + \psi^{21} + \xi^{21} \end{aligned} \tag{15}$$

where v^{21} is measured and known. Define the estimation errors as:

$$\begin{aligned} e_v^{11} =& v^{11} - \hat{v}^{11} \\ e_v^{21} =& v^{21} - \hat{v}^{21} \\ e_\delta^{11} =& \delta_j^{11} - \hat{\delta}_j^{11} \\ e_\delta^{21} =& \delta_j^{21} - \hat{\delta}_j^{21} \end{aligned} \tag{16}$$

with $\hat{v}^{11}$ and $\hat{v}^{21}$ being the estimates of v^{11} and v^{21} defined in (18). Denote

$$\begin{aligned} \hat{\delta}_j^{11} =& (A_z^{11} - A_j^{11})\hat{v}^{11} + (A_z^{12} - A_j^{12})\hat{v}^{21} \\ &+ (B_z^{11} - B_j^{11})u \\ \hat{\delta}_j^{21} =& (A_z^{21} - A_j^{21})\hat{v}^{11} + (A_z^{22} - A_j^{22})\hat{v}^{21} \\ &+ (B_z^{21} - B_j^{21})u \end{aligned} \tag{17}$$

The following observer is proposed in [13]:

$$\begin{aligned} \dot{\hat{v}}^{11} =& A_j^{11}\hat{v}^{11} + A_j^{12}\hat{v}^{21} + B_j^{11}u + \hat{\delta}_j^{11} + A_j^{12}e_v^{21} \\ \dot{\hat{v}}^{21} =& A_j^{21}\hat{v}^{11} + A_j^{22}\hat{v}^{21} + B_j^{21}u + \hat{\delta}_j^{21} + (A_j^{22} - A_j^{s})e_v^{21} - N \end{aligned} \tag{18}$$

with $A_j^s \in R^{n_y}$ a design matrix and N and chosen as [13]:

$$N = -\alpha \|P^{21}\| P^{21^{-1}} \frac{e_v^{21}}{\|e_v^{21}\|}; \text{ if } e_v^{21} \neq 0 \tag{19}$$

and 0 otherwise; α is a positive scalar and $P^{21} \in R^{n_y}$ is the solution of the Lyapunov equation

$$A_j^{s^T} P^{21} + (*) = -Q^{21} \tag{20}$$

for a given $Q^{21} \in R^{n_y}$ and $Q^{21} = (Q^{21})^T > 0$. Note that since v^{21} is measured, e_v^{21} can be used in the observer. Using (16), the error dynamics are:

$$\begin{aligned} \dot{e}_v^{11} &= A_j^{11} e_v^{11} + e_\delta^{11} + \psi^{11} \\ \dot{e}_v^{21} &= A_j^{21} e_v^{11} + A_j^s e_v^{21} + N + e_\delta^{21} + \psi^{21} + \xi^{21} \end{aligned} \tag{21}$$

As stated by [13], it is assumed that the model mismatches are upper bounded with a known value:

$$\begin{aligned} &\|e_\delta^{11}\| \leq k_2 \|e_v^{11}\| + k_3 \|e_v^{21}\| \\ &\|e_\delta^{21}\| \leq k_4 \|e_v^{11}\| + k_5 \|e_v^{21}\| \\ &\|\psi^{11}\| \leq k_6; \ \|\psi^{21}\| \leq k_7 \end{aligned} \tag{22}$$

for some $k_i, i = \overline{2,7}$. This assumption is realistic because the states of a physical process can not extend a certain physical bound, thus the upper limits can be computed. The following Lyapunov function has been used in [13] to prove the stability of the error dynamics:

$$V = e_v^{11^T} P^{11} e_v^{11} + e_v^{21^T} P^{21} e_v^{21} \tag{23}$$

where $P^{11} \in R^{n_x}, P^{11} = (P^{11})^T > 0$. Computing the derivative of (23) results in:

$$\begin{aligned} \dot{V} =& (e_v^{11})^T ((A_j^{11})^T P^{11} + (*)) e_v^{11} \\ &+ (e_v^{21})^T ((A_j^s)^T P^{21} + (*)) e_v^{21} \\ &+ (e_v^{11})^T (A_j^{21})^T P^{21} e_v^{21} + (e_v^{21})^T P^{21} A^{21} e_v^{11} \\ &+ (e_v^{21})^T P^{21} N + (*) + \big((e_v^{11})^T P^{11} (e_\delta^{11} + \psi^{11})\big) + (*) \\ &+ \big((e_v^{21})^T P^{21} (e_\delta^{21} + \psi^{21} + \xi^{21})\big) + (*) \end{aligned} \tag{24}$$

Let $P^{11} \in R^{(n_x - n_y)}, P^{11} > 0$ be a solution of the inequality:

$$A_j^{11^T} P^{11} + (*) \leq -\tilde{Q} \tag{25}$$

where $\tilde{Q} > 0$ and define $Q^{11} \in R^{(n_x - n_y)}, Q^{11} > 0$ as:

$$Q^{11} = \tilde{Q} - A_j^{21^T} P^{21} Q^{21^{-1}} P^{21} A_j^{21} \tag{26}$$

Furthermore, note that

$$\begin{aligned} &\big(e_v^{21} - (Q^{21})^{-1} P^{21} A_j^{21} e_v^{11}\big)^T Q^{21} \big(e_v^{21} - (Q^{21})^{-1} P^{21} A_j^{21} e_v^{11}\big) \\ &= (e_v^{21})^T Q^{21} e_v^{21} - (e_v^{11})^T (A_j^{21})^T P^{21} e_v^{21} - (e_v^{21})^T P^{21} A^{21} e_v^{11} \\ &+ (e_v^{11})^T (A_j^{21})^T P^{21} (Q^{21})^{-1} P^{21} A_j^{21} e_v^{11} \end{aligned} \tag{27}$$

Substituting (20) and (25) in (24) and using the notation $\tilde{e}_v^{21} = (e_v^{21} - Q^{21^{-1}} P^{21} A_j^{21} e_v^{11})$ results:

$$\begin{aligned} \dot{V} \leq& - e_v^{11^T} \tilde{Q} e_v^{11} + e_v^{11^T} A_j^{21^T} P^{21} Q^{21^{-1}} P^{21} A_j^{21} e_v^{11} \\ &- (\tilde{e}_v^{21})^T Q^{21} \tilde{e}_v^{21} + (e_v^{21})^T P^{21} N + (*) \\ &+ \big(e_v^{11^T} P^{11} (e_\delta^{11} + \psi^{11})\big) + (*) \\ &+ \big(e_v^{21^T} P^{21} (e_\delta^{21} + \psi^{21} + \xi^{21})\big) + (*) \end{aligned} \tag{28}$$

Substituting (19) and (26) and using the bound of the errors (22) gives:

$$\begin{aligned} \dot{V} \leq& - e_v^{11^T} Q^{11} e_v^{11} - (\tilde{e}_v^{21})^T Q^{21} \tilde{e}_v^{21} \\ &+ 2(-\alpha + k_6 + k_7) \|P^{21}\| \|e_v^{21}\| \\ &+ 2k_5 \|P^{11}\| \|e_v^{11}\| \\ &+ 2\big(k_1 \|P^{11}\| \|e_v^{11}\|^2 + k_2 \|P^{11}\| \|e_v^{11}\| \|e_v^{21}\|\big) \\ &+ 2\big(k_3 \|P^{21}\| \|e_v^{11}\| \|e_v^{21}\| + k_4 \|P^{21}\| \|e_v^{21}\|^2\big) \end{aligned} \tag{29}$$

which is negative if the following conditions are satisfied [13]:

$$\begin{aligned} &eig(Q^{11}) > 2k_1 \|P^{11}\| + \frac{2k_5 \|P^{11}\|}{E_v^{11}} \\ &\alpha > k_2 E_v^{11} \frac{\|P^{11}\|}{\|P^{21}\|} + k_3 E_v^{11} + k_4 E_v^{21} + k_6 + k_7 \end{aligned} \tag{30}$$

for $\|e_v^{11}\| \leq E_v^{11}$ and $\|e_v^{21}\| \leq E_v^{21}$, where E_v^{11} and E_v^{21} are known. All the computations up to this point were done in the transformed model. Returning to the main model (4) the corresponding robust observer can be expressed as:

$$\begin{aligned} \dot{\hat{x}} &= A_z \hat{x} + B_z u + \psi + G_1 (y - \hat{y}) - G_2 N \\ y &= C \hat{x} \end{aligned} \tag{31}$$

where G_1 is the gain for the linear part and G_2 is the gain for the nonlinear part and N is given in (19). The gains are computed as:

$$G_1 = T^{-1} \begin{bmatrix} A_j^{12} \\ A_j^{22} - A_j^s \end{bmatrix}; \qquad G_2 = T^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}; \tag{32}$$

Given the system (4) and the sliding observer similar (31), our goal is to design an exponentially stable observer together with fuzzy state feedback controller

$$u = -K_z \hat{x}$$

such that the closed loop system

$$\begin{aligned} \dot{x} &= A_z x + B_z (-K_z \hat{x}) + \psi + R\xi \\ &= A_z x - B_z K_z (x - e) + \psi + R\xi \\ &= (A_z - B_z K_z) x + B_z K_z e + \psi + R\xi \end{aligned} \tag{33}$$

is exponentially stable.

III. Main result

As mentioned previously, our goal is to design a fuzzy state feedback controller:

$$u = -K_z \hat{x} \tag{34}$$

that stabilizes the system (4). The closed-loop system using the controller (34), is (repeated here for convenience):

$$\dot{x} = (A_z - B_zK_z)x + B_zK_ze + \psi + R\xi \tag{35}$$

The estimation error dynamics using the observer (31) are :

$$\begin{aligned} \dot{e} =&\dot{x} - \dot{\hat{x}} \\ =&(A_z - B_zK_z)x + B_zK_ze + \psi + R\xi \\ &- A_z\hat{x} + B_zK_z\hat{x} - \psi - G_1(y - \hat{y}) + G_2N \\ =&(A_z - G_1C)e + R\xi + G_2N \end{aligned} \tag{36}$$

To develop the overall design conditions, we consider the dynamics of x and e that yield the augmented dynamics as:

$$\begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix} = \begin{bmatrix}(A_z - B_zK_z) & B_zK_z \\ 0 & (A_z - G_1C)\end{bmatrix}\begin{bmatrix}x\\ e\end{bmatrix} + \begin{bmatrix}\psi + R\xi \\ G_2N + R\xi\end{bmatrix} \tag{37}$$

We make the following notations:

$$\begin{aligned} \tilde{x} &= \begin{bmatrix}x\\ e\end{bmatrix} \\ \tilde{A} &= \begin{bmatrix}(A_z - B_zK_z) & B_zK_z \\ 0 & (A_z - G_1C)\end{bmatrix} \\ \tilde{D} &= \begin{bmatrix}\psi + R\xi \\ G_2N + R\xi\end{bmatrix} \end{aligned} \tag{38}$$

thus (37) can be rewritten as:

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{D} \tag{39}$$

To develop the conditions for the stability of the closed-loop system a simple quadratic Lyapunov function is considered:

$$V = x^T\tilde{P}^{11}x + e^TP_{21}e \tag{40}$$

We split (40) as $V_1 = x^T\tilde{P}^{11}x$ and $V_2 = e^TP_{21}e$. From the developments in Section II we have $e = T^{-1}e_v$ and we denote $\tilde{P}^{21} = T^{-T}P_{21}T^{-1}$. Then V_2 can be expressed as $V_2 = (e_v)^T\tilde{P}^{21}e_v$. If the conditions in (30) are satisfied then it can be easily concluded that $\dot{V}_2$ is negative. Next we compute the derivative of V_1 as follows:

$$\begin{aligned} \dot{V}_1 =&\dot{x}(t)^T\tilde{P}^{11}x(t) + x(t)^T\tilde{P}^{11}\dot{x}(t) \\ =&x^T\big(\tilde{P}^{11}(A_z - B_zK_z) + (*)\big)x + x^T\tilde{P}^{11}B_zK_ze + (*) \\ &+ x^T\tilde{P}^{11}(\psi + R\xi) + (*) \end{aligned} \tag{41}$$

To develop conditions for the asymptotic stability of the system (37), we first develop the design conditions for the observer such that instead of asymptotic, exponential stability of the estimation error is ensured. Thus, N is selected such that it compensate for the terms that multiplies e_v^{21}. Starting from (28) and using the bounding of the errors in (22) gives:

$$\begin{aligned} \dot{V}_2 \leq& -(e_v^{11})^TQ^{11}e_v^{11} + 2k_2(e_v^{11})^TP^{11}e_v^{11} - (\tilde{e}_v^{21})^TQ^{21}\tilde{e}_v^{21} \\ \leq& -(e_v^{11})^T(Q^{11} - 2k_2P^{11})e_v^{11} - (\tilde{e}_v^{21})^TQ^{21}\tilde{e}_v^{21} \end{aligned} \tag{42}$$

The goal is to obtain $\dot{V}_2 \leq -e_v^T\lambda Ie_v$, for some $\lambda > 0$. Substituting back $\tilde{e}_v^{21}$ into (42) one obtains:

$$\begin{aligned} \dot{V}_2 \leq& -(e_v^{11})^T(Q^{11} - 2k_2P^{11})e_v^{11} \\ &- (e_v^{21} - {Q^{21}}^{-1}P^{21}A_j^{21}e_v^{11})^TQ^{21}(e_v^{21} - {Q^{21}}^{-1}P^{21}A_j^{21}e_v^{11}) \\ &\leq -e_v^T\lambda Ie_v \end{aligned} \tag{43}$$

Rearranging (43) results in:

$$\begin{aligned} &-(e_v^{11})^T(Q^{11} - 2k_2P^{11} - \lambda I)e_v^{11} \\ &-(e_v^{21})^T(Q^{21} - \lambda I)(e_v^{21}) + (e_v^{21})^TP^{21}A_j^{21}e_v^{11} + (*) \\ &-(e_v^{11})^T{A_j^{21}}^TP^{21}(Q^{21})^{-1}P^{21}A_j^{21}e_v^{11} \leq 0 \end{aligned} \tag{44}$$

We rewrite (44) in a matrix form as:

$${e_v}^T\begin{bmatrix}\Gamma_{11}^{45} & {A_j^{21}}^TP^{21} \\ (*) & Q^{21} - \lambda I\end{bmatrix}e_v \geq 0 \tag{45}$$

with

$$\Gamma_{11}^{45} = Q^{11} - 2k_2P^{11} - \lambda I + {A_j^{21}}^TP^{21}(Q^{21})^{-1}P^{21}A_j^{21}$$

Since $Q^{11} = \tilde{Q} - {A_j^{21}}^TP^{21}{Q^{21}}^{-1}P^{21}A_j^{21}$, $\Gamma_{11}^{45} = \tilde{Q} - 2k_2P^{11} - \lambda I$, (45) is satisfied if:

$$\begin{bmatrix}\tilde{Q} - 2k_2P^{11} - \lambda I & {A_j^{21}}^TP^{21} \\ (*) & Q^{21} - \lambda I\end{bmatrix} \geq 0 \tag{46}$$

Similarly to the developments in Section II, if the following conditions are satisfied:

$$\begin{aligned} &\alpha > k_2E_v^{11}\frac{\|P^{11}\|}{\|P^{21}\|} + k_3E_v^{11} + k_4E_v^{21} + k_6 + k_7 \\ &\begin{bmatrix}\tilde{Q} - 2k_2P^{11} - \lambda I & {A_j^{21}}^TP^{21} \\ (*) & Q^{21} - \lambda I\end{bmatrix} \geq 0 \\ &{A_j^{11}}^TP^{11} + (*) \leq -\tilde{Q} \\ &{A_j^{s}}^TP^{21} + (*) \leq -Q^{21} \end{aligned} \tag{47}$$

then $\dot{V}_2 \leq -e_v^T\lambda Ie_v$. Computing the derivative of (40), replacing $\dot{V}_1$ with (41) and taking into account that $\dot{V}_2 \leq -e_v^T\lambda Ie_v$, one obtains:

$$\begin{aligned} \dot{V} \leq& x^T\big(\tilde{P}^{11}(A_z - B_zK_z) + (*)\big)x + x^T\tilde{P}^{11}B_zK_ze + (*) \\ &+ x^T\tilde{P}^{11}(\psi + R\xi) + (*) - e^TT^T\lambda ITe \end{aligned} \tag{48}$$

Using *Property 1* for some $\epsilon > 0$, we have:

$$\begin{aligned} x^T\tilde{P}^{11}(\psi + R\xi) + (*) \leq& x^T\tilde{P}^{11}\frac{1}{\epsilon}\tilde{P}^{11}x \\ &+ (\psi + R\xi)^T\epsilon(\psi + R\xi) \end{aligned} \tag{49}$$

Assuming that $\|\psi + R\xi\| \leq k_8\|x\|$ gives:

$$x^T\tilde{P}^{11}(\psi + R\xi) + (*) \leq x^T\tilde{P}^{11}\frac{1}{\epsilon}\tilde{P}^{11}x + k_8^2x^Tx\epsilon \tag{50}$$

Using the notation (38) together with the condition (50), $\dot{V}$ in (48) is negative, if:

$$\tilde{x}^T\begin{bmatrix}\Gamma_{11}^{51} & \tilde{P}^{11}B_zK_z \\ (*) & -T^T\lambda T\end{bmatrix}\tilde{x} < 0 \tag{51}$$

with

$$\Gamma_{11}^{51} = \tilde{P}^{11}(A_z - B_z K_z) + (*) + \tilde{P}^{11}\frac{1}{\epsilon}\tilde{P}^{11} + k_8^2\epsilon$$

We multiply λ with $\lambda\lambda^{-1}$ and apply congruence with diag$((H,\, H)$ where $H = (\tilde{P}^{11})^{-1}$, that results in:

$$\begin{bmatrix} \Gamma_{11}^{52} & B_z K_z H \\ (*) & -HT^T\lambda\lambda^{-1}\lambda TH \end{bmatrix} < 0 \tag{52}$$

with

$$\Gamma_{11}^{52} = (A_z - B_z K_z)H + (*) + \frac{1}{\epsilon}I + Hk_8^2\epsilon H$$

Applying the Schur complement on $Hk_8^2\epsilon H$ and using *Property 2* one obtains:

$$\begin{bmatrix} \Gamma_{11}^{53} & B_z K_z H & Hk_8 \\ (*) & -\lambda HT^T + \lambda TH + \lambda I & 0 \\ (*) & (*) & -\frac{1}{\epsilon}I \end{bmatrix} < 0 \tag{53}$$

with

$$\Gamma_{11}^{53} = (A_z - B_z K_z)H + (*) + \frac{1}{\epsilon}IB_z K_z H$$

We denote $S_z = K_z H$ resulting in:

$$\begin{bmatrix} \Gamma_{11}^{54} & B_z S_z & Hk_8 \\ (*) & -\lambda HT^T + \lambda TH + \lambda I & 0 \\ (*) & (*) & -\frac{1}{\epsilon}I \end{bmatrix} < 0 \tag{54}$$

where

$$\Gamma_{11}^{54} = (A_z H - B_z S_z) + (*) + \frac{1}{\epsilon}I$$

(54) holds if

$$\frac{2}{s-1}F_{ii} + F_{il} + F_{li} \leq 0 \quad \forall i, l = 1, ..., s \tag{55}$$

where:

$$F_{il} = \begin{bmatrix} \Gamma_{il}^{56} & B_i S_l & Hk_8 \\ (*) & -\lambda HT^T + \lambda TH + \lambda I & 0 \\ (*) & (*) & -\frac{1}{\epsilon}I \end{bmatrix} \tag{56}$$

and

$$\Gamma_{il}^{56} = (A_i H - B_i S_l) + (*) + \frac{1}{\epsilon}I$$

The results are summarised in the following theorem.

Theorem: Consider system (1) with the corresponding fuzzy model (2) and linear dominant approximations (5), where the model mismatches are bounded as:

$$\begin{aligned} &\|\xi\| \leq k_1\|y\| \\ &\|e_\delta^{11}\| \leq k_2\|e_v^{11}\| + k_3\|e_v^{21}\| \\ &\|e_\delta^{21}\| \leq k_4\|e_v^{11}\| + k_5\|e_v^{21}\| \\ &\|\psi^{11}\| \leq k_6;\ \|\psi^{21}\| \leq k_7 \\ &\|\psi + R\xi\| \leq k_8\|x\| \\ &\|e_v^{11}\| \leq E_v^{11} \\ &\|e_v^{21}\| \leq E_v^{21} \end{aligned}$$

If there exists $P^{11} = P^{11^T} > 0, P^{21} = P^{21^T} > 0, Q^{21} = (Q^{21})^T > 0, \tilde{Q} = \tilde{Q}^T > 0, S, H = H^T > 0, \lambda > 0, \epsilon > 0$ such that:

$$\begin{aligned} &\begin{bmatrix} \tilde{Q} - 2k_2 P^{11} - \lambda I & A_j^{21^T} P^{21} \\ (*) & Q^{21} - \lambda I \end{bmatrix} \geq 0 \\ &A_j^{11^T} P^{11} + (*) \leq -\tilde{Q} \\ &A_j^{s^T} P^{21} + (*) \leq -Q^{21} \\ &\alpha > k_2 E_v^{11}\frac{\|P^{11}\|}{\|P^{21}\|} + k_3 E_v^{11} + k_4 E_v^{21} + k_6 + k_7 \\ &\frac{2}{s-1}F_{ii} + F_{il} + F_{li} \leq 0 \quad \forall i, l = 1, ..., s \end{aligned} \tag{57}$$

where

$$\begin{aligned} &F_{il} = \begin{bmatrix} \Gamma_{il}^{58} & B_i S_l & Hk_8 \\ (*) & -\lambda HT^T + \lambda TH + \lambda I & 0 \\ (*) & (*) & -\frac{1}{\epsilon} \end{bmatrix} \\ &\Gamma_{il}^{58} = (A_i H - B_i S_l) + (*) + \frac{1}{\epsilon}I \end{aligned} \tag{58}$$

then the closed loop system (35) is asymptotically stabilized by the controller (34) using the states estimated by the observer (31).

Remark: Note that if (45) holds, then the condition $\tilde{Q} - A_j^{21^T} P^{21} Q^{21^{-1}} P^{21} A_j^{21} > 0$ is satisfied.

IV. Example

In this section we illustrate the performances of the proposed method on the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{3 + \sin(x_2)}{2}x_2 \\ \dot{x}_2 &= \frac{3 - \sin(x_2)}{2}x_1 + 6x_2 - 2\sin(x_2)x_2 \\ &\quad + 2u - \sin(x_2)u + \xi \\ y &= x_2 \end{aligned} \tag{59}$$

where $\|\xi\| \leq 0.5\|y\|$. For the simulations we use $\xi = 0.5x_2$. Note that this system is open-loop unstable. The corresponding fuzzy model is :

$$\dot{x}(t) = A_z x(t) + B_z u(t) + \psi + R\xi \tag{60}$$

where

$$\begin{aligned} A_{11} &= \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix},\ A_{21} = \begin{bmatrix} -1 & 1 \\ 2 & 8 \end{bmatrix}, \\ B_{11} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_{21} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \\ h_1(z) &= \frac{sin(z) + 1}{2},\ h_2(z) = 1 - h_1(z),\ z = x_2. \end{aligned} \tag{61}$$

Since the model has been obtained using the sector nonlinearity approach, $\psi = 0$. The linear dominant model is considered to be:

$$A_j = \frac{A_{11} + A_{21}}{2},\ B_j = \frac{B_{11} + B_{21}}{2} \tag{62}$$

The bounds on the errors are: $k_1 = 0.5$, $k_2 = 1$, $k_3 = k_5 = 1$, $k_4 = 1$, $k_6 = k_7 = 1.5$, $k_8 = 1$, $\alpha = 16$, $E_v^{11} = 1.5$, $E_v^{21} = 3$, and they are verified a posteriori. Choosing $A_s^{22} = -5$, $\lambda = 10$ and solving (58) we obtain the observer gains

$$G_1 = \begin{bmatrix} 1.5 \\ 11.5 \end{bmatrix}; \; G_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{63}$$

and controller gains

$$K_1 = \begin{bmatrix} 3.21 & 9.20 \end{bmatrix}; \; K_2 = \begin{bmatrix} 1.87 & 6.23 \end{bmatrix} \tag{64}$$

Using these gains, the nonlinear system is stabilized, as can be seen in Figure 1. For this particular trajectory, the initial condition was $x(0) = [1\,0.5]^T$, while the estimated states were initialized at $\hat{x}(0) = [0\,0]^T$.

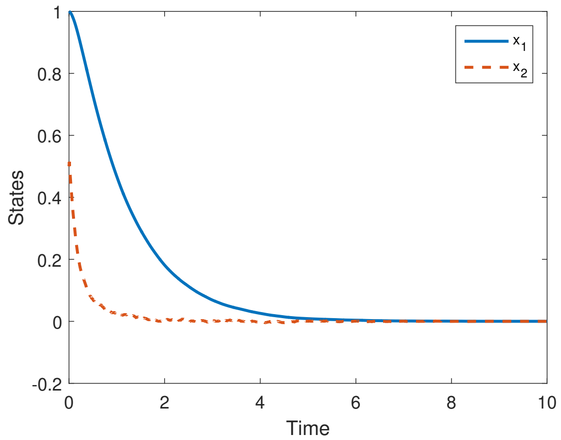

Fig. 1. States of the controlled nonlinear system

V. Conclusions

This paper focuses on the robust sliding mode observer-based fuzzy control for a nonlinear system. A linear dominant local model is considered, based on which a robust observer is designed. The uncertainties that appear are also included in the design. Next, a fuzzy state-feedback controller is designed. The observer-based controller has been illustrated on a numerical example.

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