Generalized LMI observer design for discrete-time nonlinear descriptor models

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***Abstract* :**

The present paper provides a systematic way to generalize Takagi-Sugeno observer design for discrete-time nonlinear descriptor models. The approach is based on Finsler's lemma, which decouples the observer gains from the Lyapunov function. The results are expressed as strict LMI constraints. To obtain more degrees of freedom without altering the number of LMI constraints and thus relax the conditions, delayed Lyapunov functions and delayed observer gains are considered. Even more relaxed results are developed by extending the approach to α-sample variation. The effectiveness of the proposed methods is illustrated via examples.

***Keywords* :**

Nonlinear descriptor model, observer design, linear matrix inequality, non-quadratic Lyapunov function.

1. Introduction

A large family of nonlinear models can be represented as Takagi-Sugeno (TS) models [1]. Several methods to obtain a TS representation exist; the most common are via linearization in several operational points [2] and using the sector nonlinearity approach [3]. During the last years, the sector nonlinearity approach has been employed since the resulting TS model exactly represents the original nonlinear model in a compact set of the state space. A TS model is a collection of linear models interconnected by membership functions (MFs), which are nonlinear and hold the convex sum property [4]. The analysis of TS models is performed through the direct Lyapunov method and one of the main goals is to express the conditions in terms of linear matrix inequalities (LMIs) [5], [6]. Using the sector nonlinearity, the number of linear models (vertices) exponentially increases with the number of nonlinearities in the original model. For example, mechanical systems can involve a high number of states and numerous nonlinearities, thus resulting in a standard TS representation with a large number of rules, increasing the computational cost in a way that it can be difficult to handle with the actual LMI solvers [4], [7], [8].

Since the pioneering results of the non-quadratic approach [9], the analysis and design conditions for discrete-time TS models have witnessed interesting improvements [9]–[14]. Recently, a non-quadratic Lyapunov function using past samples in its MFs has been proposed in [15] for the observer design and generalized for state feedback controller design in [16].

For systems represented via nonlinear descriptor models [17], an interesting way to handle them has been given in [18]: a TS descriptor representation. This extension of TS models arises when applying twice the sector nonlinearity methodology: once for the right-hand side of the equation and another for the left-hand side. Generally, a TS descriptor model reduces the number of linear models and also the number of LMI constraints with respect to standard ones [8], [19]–[21]. Moreover, the so-called descriptor redundancy [22] has been used to obtain relaxed conditions for those models that do not appear in a natural descriptor form [23]–[25]. The motivation of the work is twofold. The first one is considering that numerous models, for example in the mechanical field [8] [19] [20], do belong naturally to this family of models, their study appeals specific tools. The second is, not only to propose LMI constraints solutions, but also to derive, depending on some complexity parameter, conditions that are less and less conservative.

When the state vector is not fully available an observer has to be implemented. The observer design for descriptor models has been discussed in [26]–[29]. For nonlinear systems with a constant rank-deficient descriptor matrix, few results exist that involve LMI conditions [30], [31]. Moreover, existing results include restrictions such as linear output matrices, i.e., linear measurements.

Previous results on descriptor models only consider a constant rank-deficient descriptor matrix. This paper develops conditions for the observer design for nonlinear descriptors with a non-constant full-rank descriptor matrix. In a sense, it extends the standard TS observer results [7], [32], [33] to the TS descriptor framework. LMI conditions are obtained via non-quadratic Lyapunov functions and Finsler's lemma. Finsler's lemma is used to avoid the explicit substitution of the closed-loop dynamics of the estimation error [34] and to decouple the Lyapunov function matrices from the observer gains [15], [21]. Furthermore, the paper provides a general framework which encompasses previous results for observer design for discrete-time descriptor models, both linear and TS. At last, the discrete nature of the Lyapunov function via α-sample variation is exploited as in [11] to obtain results whose conservativeness decreases according to a complexity parameter, i.e. α the number of past samples considered in the Lyapunov function.

The paper is divided as follows: Section 2 introduces the discrete-time TS descriptor model, provides useful notations, and motivates this research via an example; Section 3 presents and discusses the main results on the observer design and illustrates them; Section 4 extends the results using α-sample variation; Section 5 concludes the paper and gives some perspectives.

2. Preliminaries

2.1 TS descriptor models

Consider the following discrete-time nonlinear descriptor model:

$$\begin{aligned} E(x_k)x_{k+1} &= A(x_k)x_k + B(x_k)u_k \\ y_k &= C(x_k)x_k, \end{aligned} \tag{1}$$

where $x_k \in$ □ is the state vector, $u_k \in$ □ is the control input vector, $y_k \in$ □ is the output vector, and k is the current sample. Matrices $A(x_k)$, $B(x_k)$, $C(x_k)$, and $E(x_k)$ are assumed to be smooth in a compact set Ω_x of the state space including the origin. Moreover, $E(x_k)$ is assumed to be a regular matrix, at least in the compact set Ω_x. In what follows, arguments will be omitted when they can be easily inferred. An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left-hand side, i.e.,

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & C \end{bmatrix}, \qquad A + B + A^T + B^T + C = A + B + (*) + C.$$

Using the sector nonlinearity approach [4], the p_a nonlinear terms in the right-hand side of (1) are captured via the membership functions (MFs) $h_i(z)$, $i \in \{1,2,\ldots,2^{p_a}\}$. The p_e nonlinear terms in the left-hand side of (1) are grouped in MFs $v_j(z)$, $j \in \{1,2,\ldots,2^{p_e}\}$. These MFs hold the convex sum property in the compact set Ω_x, i.e., $\sum_{i=1}^{r_a} h_i(z) = 1$, $h_i(z) \geq 0$, $\sum_{j=1}^{r_e} v_j(z) = 1$, $v_j(z) \geq 0$ with $r_a = 2^{p_a}$ and $r_e = 2^{p_e}$. In this work, the MFs depend on the premise variables grouped in the vector $z \in \square$, $p = p_a + p_e$, which is assumed to be known [7].

Using the methodology stated above, from the nonlinear model (1) an exact TS descriptor model is obtained [19]:

$$\begin{aligned} \sum_{j=1}^{r_e} v_j(z) E_j x_{k+1} &= \sum_{i=1}^{r_a} h_i(z)\left(A_i x_k + B_i u_k\right) \\ y_k &= \sum_{i=1}^{r_a} h_i(z) C_i x_k, \end{aligned} \tag{2}$$

where matrices A_i, B_i, and C_i, $i \in \{1,2,\ldots$ represent the *i*-th linear right-hand side model (2) and E_j, $j \in \{1,2,\ldots$ represent the *j*-th left-hand side model of the TS descriptor model. The premise vector is assumed to be available in time; it does not have to be estimated.

2.2 Properties and lemmas

Generally in the TS-LMI framework, it is natural to obtain inequality conditions involving convex sums, for instance:

$$\sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} h_{i_1}(z(k)) h_{i_2}(z(k)) \Upsilon_{i_1 i_2} < 0, \tag{3}$$

where $\Upsilon_{i_1 i_2} = \Upsilon_{i_1 i_2}^T$ $i_i, i_2 \in \{1,2,\ldots,\mathrm{r}_a\}$. In order to obtain LMI conditions, the MFs must be removed from (3). Throughout this paper, the following sum relaxation scheme will be employed.

Lemma 1. [35] The double convex-sum (3) is negative if

$$\Upsilon_{i_i i_i} < 0, \quad \forall i_i \in \{1,2,\ldots,\mathrm{r}_a\},$$
$$\frac{2}{r_a - 1}\Upsilon_{i_i i_i} + \Upsilon_{i_i i_2} + \Upsilon_{i_2 i_i} < 0, \quad i_i, i_2 \in \{1,2,\ldots,\mathrm{r}_a\}, \quad i_1 \neq i_2, \tag{4}$$

hold.

Note that Lemma 1 is one the possible schemes to drop off the convex MFs from (3). Other schemes that include slack variables [36], [37] exist in the literature and they apply directly on the results presented in this work.

Due to the different sets of MFs coming from the descriptor form, two different pairs of convex sums may appear, i.e.,

$$\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) v_{j_1}\left(z(k)\right) v_{j_2}\left(z(k)\right) \Upsilon_{i_1 i_2}^{j_1 j_2} < 0, \tag{5}$$

where $\Upsilon_{i_1 i_2}^{j_1 j_2} = \left(\Upsilon_{i_1 i_2}^{j_1 j_2}\right)^T$, $i_i, i_2 \in \{1,2,\ldots,\mathrm{r}_a\}$, $j_1, j_2 \in \{1,2,\ldots$. Therefore, an extension of Lemma 1 follows.

Lemma 2. Sufficient condition for (5) to hold are

$$\begin{cases} \Upsilon_{i_1 i_1}^{j_1 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, j_1 \in \{1,2,\ldots,r_e\}, \\ \dfrac{2}{r_e - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, \quad j_1 \neq j_2, \\ \dfrac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1} < 0, \quad \forall j_1 \in \{1,2,\ldots,r_3\}, \quad i_1 \neq i_2, \\ \dfrac{4}{(r_e - 1)(r_a - 1)}\Upsilon_{i_1 i_1}^{j_1 j_1} + \dfrac{2}{r_e - 1}\left(\Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1}\right) + \dfrac{2}{r_a - 1}\left(\Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1}\right) \\ \quad + \Upsilon_{i_1 i_2}^{j_1 j_2} + \Upsilon_{i_2 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_2}^{j_2 j_1} + \Upsilon_{i_2 i_1}^{j_2 j_1} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\}, j_1, j_2 \in \{1,2,\ldots,r_e\}, \ i_1 \neq i_2, \quad j_1 \neq j_2. \end{cases} \tag{6}$$

Proof. See appendix A.

Lemma 3. [34] (Finsler's lemma). Let $\mathrm{X} \in \square$ $\mathrm{Q} = \mathrm{Q}^T \in \square$, and $\mathrm{W} \in \square$ such that $rank(\mathrm{W}) < n$; the following expressions are equivalent:

a) $\chi^T \mathrm{Q} \chi < 0, \quad \forall \chi \in \{\chi \in \square \quad , \mathrm{W}\chi = 0\}$.

b) $\exists \mathrm{M} \in \square \quad \mathrm{M}\ \mathrm{W} + \mathrm{W}^T \mathrm{M}^T + \mathrm{Q} < 0$.

2.3 $E(x)$ invertible: motivation

In this work, $E(x) = \sum_{j=1}^{r_e} v_j(z) E_j$ is assumed to be a regular matrix. This is motivated as follows

1) This case appears in many cases, for example for mechanical systems, matrix $E(x)$ contains the inertia matrix and is therefore regular [8], [19].

2) Since the matrix $E(x)$ is regular (invertible), from the descriptor model (1) a standard state-space form can be computed:

$$x_{k+1} = E^{-1}(x)\left(A(x)x_k + B(x)u_k\right), \tag{7}$$

For a standard TS representation of (7), *classical* results can be directly applied. The nonlinear representations (1) and (7) are equivalent in the compact set of interest Ω_x; however, a TS representation from (7) may have some drawbacks: a) the term $E^{-1}(x)$ may introduce a more complex structure in $A(x)$ and $B(x)$; furthermore, if the input matrix is constant, once multiplied by $E^{-1}(x)$, it yields in $\bar{B}(x) = E^{-1}(x)B$, i.e., it is not longer constant. b) The number of vertices may increase; this could lead to numerical problems when using LMI solvers.

Therefore, the motivation of keeping the structure (1) instead of (7) is to propose solutions via LMI framework that reduces the conservatism. Previous works in the continuous case have shown such behaviour [8], [18]. Let us begin with an example together with the recent results for standard TS representations (7) [9], [15].

Example 1. Consider the discrete-time nonlinear descriptor system (1) with matrices defined as:

$$E(x) = \begin{bmatrix} 2 & -1/\left(1+x_1^2\right) \\ 1/\left(1+x_1^2\right) & 1 \end{bmatrix}, \quad A(x) = \begin{bmatrix} \cos(x_1) & -1 \\ 0.7 & -1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } C(x) = \begin{bmatrix} \sin(x_1)/x_1 \\ 0.2 \end{bmatrix}^T.$$

Note that $\det(E(x)) = \left(3+4x_1^2+2x_1^4\right)\left(1+x_1^2\right)^{-2} \neq 0$, i.e., $E(x)$ is regular for all $x \in \mathbb{R}^n$. Thus, the inverse of the descriptor matrix gives $E^{-1}(x) = \dfrac{1}{\det(E(x))}\begin{bmatrix} 1 & \eta \\ -\eta & 2 \end{bmatrix}$, $\eta = 1/\left(1+x_1^2\right)$; this means that four different nonlinearities have to be considered, which results in $r = 16$. The state variable x_1 is available. Consider the observer design problem using the Lyapunov function $V(e_k) = e_k^T \mathrm{P} e_k$, where e_k is the estimation error. Conditions using one convex sum for the Lyapunov function, i.e., $V(e_k)$ with $\mathrm{P} = \sum_{i=1}^r h_i(z(k))P_i$, which corresponds to the non-quadratic (NQ) approach in [9], rely in solving the following constraints:

$$\sum_{i_1=1}^r \sum_{i_2=1}^r \sum_{i_x=1}^r h_{i_1}(z(k))h_{i_1}(z(k))h_{i_x}(z(k+1))\begin{bmatrix} -P_{i_2} & (*) \\ G_{i_2}A_{i_1} - L_{i_2}C_{i_1} & -G_{i_2} - G_{i_2}^T + P_{i_x} \end{bmatrix} < 0. \tag{8}$$

Using another powerful non quadratic Lyapunov function, the so-called delayed non-quadratic (DNQ) one [15], i.e., $\mathrm{P} = \sum_{i_x=1}^r h_{i_x}(z(k-1))P_{i_x}$ corresponds to solving the LMI conditions issued from:

$$\sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\sum_{i_x=1}^{r} h_{i_1}\left(z(k)\right) h_{i_2}\left(z(k)\right) h_{i_x}\left(z(k-1)\right) \begin{bmatrix} -P_{i_x} & (*) \\ G_{i_2 i_x} A_{i_1} - L_{i_2 i_x} C_{i_1} & -G_{i_2 i_x} - G_{i_2 i_x}^T + P_{i_2} \end{bmatrix} < 0. \tag{9}$$

None of the two conditions above have provided (using Lemma 1 for obtaining LMIs) a feasible solution for the standard TS model computed from (7). Both sets of conditions involve three convex sums and need to solve $r^3 + r = 4112$ LMIs, where r represents the number of rules for standard TS models. Although these results, even if not conclusive since another TS representation may lead to feasible solutions, exhibits clearly the problem: inverting matrix $E(x)$ and following standard way of doing leads to unfeasibility. As proposed through this work, a natural option is to keep the descriptor structure within the TS framework, even if $E(x)$ is regular. This example will be resumed later on. ♦

Remark 1. There are recent approaches for discrete-time standard TS models based on multi-instant homogenous matrix polynomials [38]–[40] generalizing [15]. Nevertheless, their complexity (the number of LMI constraints and slack variables) is, of course, higher and they lead very quickly to numerical problems with actual LMI solvers for the system in Example 1. Therefore, Example 1 shows the importance of developing specific LMI conditions for TS descriptor models.

For sake of brevity and clarity, the notation from [16] will be used throughout this paper.

Definition 1. (Multiple sum) A multiple sum with n_{Υ_h} terms and delays evaluated at sample k of the form

$$\Upsilon_{H_0^{\Upsilon}} := \sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\cdots\sum_{i_{n_{\Upsilon_h}}=1}^{r_a} \text{[illegible]} \cdots \text{[illegible]}$$

where H_0^{Υ} is the multiset of delays $H_0^{\Upsilon} = \{d_1, d_2, \ldots \text{[illegible]}\}$.

Definition 2. (Multiset of delays) H_0^{Υ} denotes the multiset containing the delays in the multiple sum involving Υ at sample k. H_α^{Υ} denotes the multiset containing the delays in the sum Υ at sample $k+\alpha$.

Definition 3. (Cardinality) The cardinality of a multiset H_0^{Υ}, $\left|H_0^{\Upsilon}\right| = n_{\Upsilon_h}$, is defined as the number of elements in H_0^{Υ}.

Definition 4. (Index set) The index set of a multiple sum Υ_H is $\mathbf{I}_H = \{i_s : i_s = 1, 2, \ldots \text{[illegible]} \ldots \text{[illegible]}\}$, the set of all indices that appear in the sum. An element $\boldsymbol{i}$ is a multiindex.

Definition 5. (Multiplicity) The multiplicity of an element x in a multiset H, $\mathbf{1}_H(x)$ denotes the number of times this element appears in the multiset H.

Definition 6. (Union) The union of two multisets H_A and H_B is $H_C = H_A \cup H_B$ such that $\forall x \in H_C,\ \ \mathbf{1}_{H_C}(x) = \max\left\{\mathbf{1}_{H_A}(x), \mathbf{1}_{H_B}(x)\right\}$.

Definition 7. (Intersection) The intersection of two multisets H_A and H_A is $H_C = H_A \cap H_B$ such that $\forall x \in H_C$, $\mathbf{1}_{H_C}(x) = \min\{\mathbf{1}_{H_A}(x), \mathbf{1}_{H_B}(x)\}$.

Definition 8. (Sum) The sum of two multisets H_A and H_B is $H_C = H_A \oplus H_B$ such that $\forall x \in H_C$, $\mathbf{1}_{H_C}(x) = \mathbf{1}_{H_A}(x) + \mathbf{1}_{H_B}(x)$.

Definition 9. (Projection of an index) The projection of the index $\boldsymbol{i} \in \mathbf{I}_{H_A}$, to the multiset of delays H_B, $pr_{H_B}^{\boldsymbol{i}}$ is the part of the index that corresponds to the delays in $H_A \cap H_B$.

Example 2. Consider the following multiple convex sum:

$$\Upsilon_{H_0^{\Upsilon}} = \sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_3=1}^{r_a}\sum_{i_4=1}^{r_a}\sum_{i_5=1}^{r_a} h_{i_1}(z(k-1))\, h_{i_2}(z(k))\, h_{i_3}(z(k+1))\, h_{i_4}(z(k+1))\, h_{i_5}(z(k+3))\, \Upsilon_{i_1 i_2 i_3 i_4 i_5}. \tag{10}$$

Then the delays contained by the multiset H_0^{Υ} in (10) are given by $H_0^{\Upsilon} = \{-1,0,1,1,3\}$ or $H_\alpha^{\Upsilon} = \{\alpha-1, \alpha, \alpha+1, \alpha+1, \alpha+3\}$. The cardinality of H_0^{Υ} is $|H_0^{\Upsilon}| = 5$. The set of the multiple sum (10) is $\mathbf{I}_H = \{i_s : i_s = 1,2,\ldots$ [illegible] $\ldots$ [illegible]. The multiplicity of the elements in H_0^{Υ} is: $\mathbf{1}_H(-1) = 1$, $\mathbf{1}_H(0) = 1$, $\mathbf{1}_H(1) = 2$, and $\mathbf{1}_H(3) = 1$. Define multiset $H_A = \{-2,-2,0,1,2,3\}$; thus $H_0^{\Upsilon} \cup H_A = \{-2,-2,-1,0,1,1,2,3\}$, $H_0^{\Upsilon} \cap H_A = \{0,1,3\}$, and $H_0^{\Upsilon} \oplus H_A = \{-2,-2,-1,0,0,1,1,1,2,3,3\}$. ♦

Using the above definitions, the discrete-time TS descriptor model (2) can be expressed as

$$\begin{aligned} E_{V_0^E} x_{k+1} &= A_{H_0^A} x_k + B_{H_0^B} u_k \\ y_k &= C_{H_0^C} x_k, \end{aligned} \tag{11}$$

where the subscript denotes the dependence of the corresponding membership functions on the premise variables at the time indices in the corresponding multiset, e.g., H_0^A, H_0^B, and H_0^C stand for MFs $h_i(z)$ $i \in \{1,2,\ldots,r_a\}$; while V_0^E stands for MFs $v_j(z)$, $j \in \{1,2,\ldots,r_e\}$. Since there are no delays in the original system (2), the multisets in (11) are $V_0^E = \{0\}$, corresponding to $v_j(z(k))$, $j \in \{1,2,\ldots,r_e\}$ and $H_0^A = H_0^B = H_0^C = \{0\}$ representing $h_i(z(k))$, $i \in \{1,2,\ldots,r_a\}$.

2.4 Problem statement

The goal is to obtain conditions for observer design for TS descriptor models. The observer considered for the TS descriptor model (11) is given by:

$$\begin{aligned} E_{V_0^E} \hat{x}_{k+1} &= A_{H_0^A} \hat{x}_k + B_{H_0^B} u_k + G_{H_0^G V_0^G}^{-1} L_{H_0^L V_0^L} (y_k - \hat{y}_k) \\ \hat{y}_k &= C_{H_0^C} \hat{x}_k, \end{aligned} \tag{12}$$

where $G_{H_0^G \mathrm{V}_0^G}$ and $L_{H_0^L \mathrm{V}_0^L}$ are the observer gains to be determined. These gains include delays given by multisets H_0^G, H_0^L, V_0^G, and V_0^L. Note that these multisets must not contain positive delays, since a positive delay refers to future state variables, which are not available for computation.

Since the scheduling variables do not depend on unmeasured states, the estimation error $e_k = x_k - \hat{x}_k$ dynamics are

$$E_{\mathrm{V}_0^E} e_{k+1} = \left(A_{H_0^A} - G_{H_0^G \mathrm{V}_0^G}^{-1} L_{H_0^L \mathrm{V}_0^L} C_{H_0^C} \right) e_k. \tag{13}$$

Thus, the observer design consists in finding $L_{H_0^L \mathrm{V}_0^L}$ and $G_{H_0^G \mathrm{V}_0^G}$ such that (13) is asymptotically stable.

3. Main results

In order to design the observer (12), consider the following Lyapunov function:

$$V(e_k) = e_k^T P_{H_0^P \mathrm{V}_0^P} e_k, \quad P_{\boldsymbol{i}_0^P, \boldsymbol{j}_0^P} = P_{\boldsymbol{i}_0^P, \boldsymbol{j}_0^P}^T > 0, \qquad \boldsymbol{i} \in \mathbf{I}_{H_0^P}, \quad \boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_0^P}. \tag{14}$$

Then the following result can be stated:

Theorem 1. The estimation error dynamics in (13) are asymptotically stable if there exist $P_{\boldsymbol{i}_s^P, \boldsymbol{j}_s^P} = P_{\boldsymbol{i}_s^P, \boldsymbol{j}_s^P}^T$, $\boldsymbol{i}_s^P = pr_{H_s^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_s^P = pr_{\mathrm{V}_s^P}^{\boldsymbol{j}}$, $L_{\boldsymbol{i}_0^L, \boldsymbol{j}_0^L}$, $\boldsymbol{i}_0^L = pr_{H_0^L}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^L = pr_{\mathrm{V}_0^L}^{\boldsymbol{j}}$, and $G_{\boldsymbol{i}_0^G, \boldsymbol{j}_0^G}$, $\boldsymbol{i}_0^G = pr_{H_0^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_0^G = pr_{\mathrm{V}_0^G}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbf{I}_{H_0^\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_0^\Gamma}$, $s = 0,1$, where $H_0^\Gamma = H_0^P \cup H_1^P \cup \left(H_0^L \oplus H_0^C \right) \cup \left(H_0^G \oplus H_0^A \right)$, $\mathrm{V}_0^\Gamma = \mathrm{V}_0^P \cup \mathrm{V}_1^P \cup \mathrm{V}_0^L \cup \left(\mathrm{V}_0^G \oplus \mathrm{V}_0^E \right)$ such that

$$\begin{bmatrix} -P_{H_0^P \mathrm{V}_0^P} & (*) \\ G_{H_0^G \mathrm{V}_0^G} A_{H_0^A} - L_{H_0^L \mathrm{V}_0^L} C_{H_0^C} & -G_{H_0^G \mathrm{V}_0^G} E_{\mathrm{V}_0^E} - E_{\mathrm{V}_0^E}^T G_{H_0^G \mathrm{V}_0^G}^T + P_{H_1^P \mathrm{V}_1^P} \end{bmatrix} < 0. \tag{15}$$

Proof. The variation of the non-quadratic Lyapunov function (14) along the estimation error is

$$\begin{aligned} \Delta V(e_k) &= e_{k+1}^T P_{H_1^P \mathrm{V}_1^P} e_{k+1} - e_k^T P_{H_0^P \mathrm{V}_0^P} e_k \\ &= \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix}^T \begin{bmatrix} -P_{H_0^P \mathrm{V}_0^P} & 0 \\ 0 & P_{H_1^P \mathrm{V}_1^P} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} < 0. \end{aligned} \tag{16}$$

Note that (13) can be expressed as the equality constraint

$$\begin{bmatrix} A_{H_0^A} - G_{H_0^G \mathrm{V}_0^G}^{-1} L_{H_0^L \mathrm{V}_0^L} C_{H_0^C} & -E_{\mathrm{V}_0^E} \end{bmatrix} \begin{bmatrix} e_k \\ e_{k+1} \end{bmatrix} = 0. \tag{17}$$

Through Lemma 3, inequality (16) under constraint (17) is equivalent to finding $\mathrm{M} \in \square$ such that:

$$\mathrm{M} \begin{bmatrix} A_{H_0^A} - G_{H_0^G \mathrm{V}_0^G}^{-1} L_{H_0^L \mathrm{V}_0^L} C_{H_0^C} & -E_{\mathrm{V}_0^E} \end{bmatrix} + (*) + \begin{bmatrix} -P_{H_0^P \mathrm{V}_0^P} & 0 \\ 0 & P_{H_1^P \mathrm{V}_1^P} \end{bmatrix} < 0. \tag{18}$$

The choice $\mathrm{M} = \begin{bmatrix} 0 & G^T_{H_0^G \mathrm{V}_0^G} \end{bmatrix}^T$ yields directly (15), thus concluding the proof. ■

Example 3. Consider the multisets $H_0^G = H_0^L = \mathrm{V}_0^G = \{0,-1\}$, $\mathrm{V}_0^L = \{0,0,-1\}$, and $H_0^P = \mathrm{V}_0^P = \{-1\}$, i.e.,

$$
\begin{aligned}
P_{H_0^P \mathrm{V}_0^P} &= P_{\{-1\},\{-1\}} = \sum_{i_x=1}^{r_a}\sum_{j_x=1}^{r_e} h_{i_x}\big(z(k-1)\big) v_{j_x}\big(z(k-1)\big) P_{i_x,j_x} \\
L_{H_0^L \mathrm{V}_0^L} &= L_{\{0,-1\},\{0,0,-1\}} = \sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\big(z(k)\big) h_{i_x}\big(z(k-1)\big) v_{j_1}\big(z(k)\big) v_{j_2}\big(z(k)\big) v_{j_x}\big(z(k-1)\big) L_{i_1 i_x, j_1 j_2 j_x} \\
G_{H_0^G \mathrm{V}_0^G} &= G_{\{0,-1\},\{0,-1\}} = \sum_{i_1=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\big(z(k)\big) h_{i_x}\big(z(k-1)\big) v_{j_1}\big(z(k)\big) v_{j_x}\big(z(k-1)\big) G_{i_1 i_x, j_1 j_x}.
\end{aligned}
\tag{19}
$$

Then, recall that $\mathrm{V}_0^E = H_0^A = H_0^B = H_0^C = \{0\}$ and therefore the conditions in Theorem 1 are

$$
\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0,-1\},\{0,-1\}} A_{\{0\}} - L_{\{0,-1\},\{0,0,-1\}} C_{\{0\}} & -G_{\{0,-1\},\{0,-1\}} E_{\{0\}} - E_{\{0\}}^T G_{\{0,-1\},\{0,-1\}}^T + P_{\{0\},\{0\}} \end{bmatrix} < 0, \tag{20}
$$

or

$$
\begin{aligned}
&\sum_{i_1=1}^{r_a}\sum_{i_2=1}^{r_a}\sum_{i_x=1}^{r_a}\sum_{j_1=1}^{r_e}\sum_{j_2=1}^{r_e}\sum_{j_x=1}^{r_e} h_{i_1}\big(z(k)\big) h_{i_2}\big(z(k)\big) h_{i_x}\big(z(k-1)\big) v_{j_1}\big(z(k)\big) v_{j_2}\big(z(k)\big) v_{j_x}\big(z(k-1)\big) \times \\
&\qquad \begin{bmatrix} -P_{i_x,j_x} & (*) \\ G_{i_2 i_x, j_2 j_x} A_{i_1} - L_{i_2 i_x, j_1 j_2 j_x} C_{i_1} & -G_{i_2 i_x, j_2 j_x} E_{j_1} - E_{j_1}^T G_{i_2 i_x, j_2 j_x}^T + P_{i_2,j_2} \end{bmatrix} < 0.
\end{aligned}
\tag{21}
$$

Note that the co-negativity problem (21) involves six convex-sums. ♦

Remark 2. It is possible to consider $G_{H_0^G \mathrm{V}_0^G} = P_{H_0^P \mathrm{V}_0^P}$. This yields a classical non-PDC-like observer. However, with respect to conditions (15), the number of decision variables will be reduced while the number of LMI conditions remains the same, i.e., it will be more conservative.

Remark 3. Considering $\mathrm{V}_0^P = \mathrm{V}_0^G = \varnothing$ and $H_0^P = H_0^G = H_0^L = \mathrm{V}_0^L = \{0\}$ the conditions in Theorem 2 of [41] are recovered.

Remark 4. The total number of sums involved in (15) is $n_{H\mathrm{V}} = \left|H_0^\Gamma\right| + \left|\mathrm{V}_0^\Gamma\right|$. Moreover, considering that the system matrices do not contain delays, the maximum number of sums involved in (15) is given by $n_{H\mathrm{V}} \le 2n_{P_h} + 2n_{P_v} + n_{L_h} + n_{L_v} + n_{G_h} + n_{G_v} + 2$, where $n_{\mathrm{A}_h} = \left|H_0^{\mathrm{A}}\right|$, e.g., $n_{P_h} = \left|H_0^P\right|$.

3.1 Selecting multisets

Note that at this point no decision has been made on the multisets involved in the observer gains and in the Lyapunov function. Actually, (15) is a sufficient condition for the stabilization of the estimation error dynamics independent of the choice of the multisets. However, they should be selected such that the conditions are the least conservative. To this end, constructive steps are given.

Step 1: the system (10) does not contain delays in its MFs, i.e., $\mathcal{V}_0^E = H_0^A = H_0^B = H_0^C = \{0\}$. In order to apply multiple sums relaxations, the multisets H_0^G, H_0^L, $\mathcal{V}_0^G$, and $\mathcal{V}_0^L$ should, at least, contain $\{0\}$. Therefore introducing only one $\{0\}$ will give the minimum representation for (15):

$$\begin{bmatrix} -P_{H_0^P \mathcal{V}_0^P} & (*) \\ G_{\{0\},\{0\}} A_{\{0\}} - L_{\{0\},\{0\}} C_{\{0\}} & -G_{\{0\},\{0\}} E_{\{0\}} - E_{\{0\}}^T G_{\{0\},\{0\}}^T + P_{H_1^P \mathcal{V}_1^P} \end{bmatrix} < 0. \tag{22}$$

Now, provided that either H_0^P, $\mathcal{V}_0^P$ or H_1^P, $\mathcal{V}_1^P$ contain $\{0\}$, for H_0^P and $\mathcal{V}_0^P$, two options are possible. Notice that a positive delay is possible in $P_{H_1^P \mathcal{V}_1^P}$, as it is not part of the observer. In case $H_0^P = \mathcal{V}_0^P = \{0\}$, we obtain the Lyapunov function in [9]. The case of $H_0^P = \mathcal{V}_0^P = \{-1\}$ corresponds to the delayed Lyapunov function [15]. In this latter case (22) writes:

$$\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0\},\{0\}} A_{\{0\}} - L_{\{0\},\{0\}} C_{\{0\}} & -G_{\{0\},\{0\}} E_{\{0\}} - E_{\{0\}}^T G_{\{0\},\{0\}}^T + P_{\{0\},\{0\}} \end{bmatrix} < 0. \tag{23}$$

Step 2: when possible, complete the multisets with delays that do not increase the number of sums. For example, (23) contains three sums of the form $h(\cdot) = \sum_{i_1=1}^{r_a} \sum_{i_2=1}^{r_a} \sum_{i_x=1}^{r_a} h_{i_1}(z(k)) h_{i_2}(z(k)) h_{i_x}(z(k-1))$ and three sums of the form $v(\cdot) = \sum_{j_1=1}^{r_e} \sum_{j_2=1}^{r_e} \sum_{j_x=1}^{r_e} v_{j_1}(z(k)) v_{j_2}(z(k)) v_{j_x}(z(k-1))$. Thus, it is possible to include the delay $\{-1\}$ in each multiple sum of $G_{H_0^G \mathcal{V}_0^G}$ and $L_{H_0^L \mathcal{V}_0^L}$ while keeping the same number of sums. Moreover, since there is no product involving $L_{H_0^L \mathcal{V}_0^L}$ and $E_{\mathcal{V}_0^E}$, the MFs $v(\cdot)$ of $L_{H_0^L \mathcal{V}_0^L}$ should be chosen as $\mathcal{V}_0^L = \mathcal{V}_0^G \oplus \mathcal{V}_0^E$, thus (23) gives

$$\begin{bmatrix} -P_{\{-1\},\{-1\}} & (*) \\ G_{\{0,-1\},\{0,-1\}} A_{\{0\}} - L_{\{0,-1\},\{0,0,-1\}} C_{\{0\}} & -G_{\{0,-1\},\{0,-1\}} E_{\{0\}} - E_{\{0\}}^T G_{\{0,-1\},\{0,-1\}}^T + P_{\{0\},\{0\}} \end{bmatrix} < 0 \tag{24}$$

without increasing the number of sums.

Step 3: (24) represents the "best" option with $\{0\}$ and $\{-1\}$. Based on the previous steps a generalization to multiple delays at the same instant, for example $H_0^P = \{-1,-1,\ldots,-1\}$, $\left|H_0^P\right| = n_{P_h}$, is direct. Table 1 presents some of the various possibilities that respect the constructive steps.

Matrix	Multisets in Theorem 1
$P_{H_0^P V_0^P}$	$H_0^P = \{-1,-1,\ldots,-1\}, \quad \lvert H_0^P \rvert = n_{P_h}$
	$V_0^P = \{-1,-1,\ldots,-1\}, \quad \lvert V_0^P \rvert = n_{P_v}$
$L_{H_0^L V_0^L}$	$H_0^L = \{\underbrace{0,0,\ldots,0}_{n_{P_h}},\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}, \quad \lvert H_0^L \rvert = 2n_{P_h}$
	$V_0^L = \{0,\underbrace{0,0,\ldots,0}_{n_{P_v}},\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}, \quad \lvert V_0^L \rvert = 1+2n_{P_v}$
$G_{H_0^G V_0^G}$	$H_0^G = \{\underbrace{0,0,\ldots,0}_{n_{P_h}},\underbrace{-1,-1,\ldots,-1}_{n_{P_h}}\}, \quad \lvert H_0^G \rvert = 2n_{P_h}$
	$V_0^G = \{\underbrace{0,0,\ldots,0}_{n_{P_v}},\underbrace{-1,-1,\ldots,-1}_{n_{P_v}}\}, \quad \lvert V_0^G \rvert = 2n_{P_v}$

Table 1. How to select multisets for Theorem 1.

Following the steps provided above, the number of sums in (15) is $n_{HV} = 2n_{P_h} + 2n_{P_v} + 2$. Table 2 states the complexity (number of scalar decision variables, number of LMIs, and number of rows in the LMI problem) of the proposed approach as well as some issued from the literature.

Approach	Number of scalar decision variables	Number of LMIs	Row size of the LMI
NQ in [9]	$0.5n_x(n_x+1)r + n_x^2 r + n_x n_y r$	$r^3 + r$	$n_x r + 2n_x r^3$
DNQ in [15]	$0.5n_x(n_x+1)r + n_x^2 r^2 + n_x n_y r^2$	$r^3 + r$	$n_x r + 2n_x r^3$
Theorem 1	$0.5n_x(n_x+1) r_a^{n_{P_h}} r_e^{n_{P_v}} + n_x^2 r_a^{n_{G_h}} r_e^{n_{G_v}} + n_x n_y r_a^{n_{L_h}} r_e^{n_{L_v}}$	$r_a^{n_{\Gamma_h}} r_e^{n_{\Gamma_v}} + r_a^{n_{P_h}} r_e^{n_{P_v}}$	$n_x r_a^{n_{P_h}} r_e^{n_{P_v}} + 2n_x r_a^{n_{\Gamma_h}} r_e^{n_{\Gamma_v}}$
Theorem 1 using the choices in Table 1	$0.5n_x(n_x+1) r_a^{n_{P_h}} r_e^{n_{P_v}} + n_x^2 r_a^{2n_{P_h}} r_e^{2n_{P_v}} + n_x n_y r_a^{2n_{P_h}} r_e^{(2n_{P_v}+1)}$	$r_a^{(2n_{P_h}+1)} r_e^{(2n_{P_v}+1)} + r_a^{n_{P_h}} r_e^{n_{P_v}}$	$n_x r_a^{n_{P_h}} r_e^{n_{P_v}} + 2n_x r_a^{(2n_{P_h}+1)} r_e^{(2n_{P_v}+1)}$

Table 2. Number of scalar decision variables, number of LMIs, and row size of the LMI for several approaches.

In what follows, we illustrate the proposed conditions on Example 1. Recall that state-of the art methods in the literature did not provide a feasible result.

Example 1 (continued). Considering the compact set $\Omega_x = \{x \in \square$ ˋ ˋ , then keeping the descriptor structure and computing an exact TS descriptor models gives in the left-hand side $r_e = 2$ due to $1/(1+x_1^2)$. In the right-hand side we have: $r_a = 4$ due to the terms $\cos(x_1)$ and $\sin(x_1)/x_1$. The constant matrices are $E_1 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A_1 = A_2 = \begin{bmatrix} 1 & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} -1 & -1 \\ 0.7 & -1.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = C_3 = [1 \quad 0.2]$, and $C_2 = C_4 = [-0.2167 \quad 0.2]$. The MFs are defined as: $v_1 = 1/(1+x_1^2)$, $v_2 = 1 - v_1$, $h_1 = \omega_0^1\omega_0^2$, $h_2 = \omega_0^1\omega_1^2$, $h_3 = \omega_1^1\omega_0^2$, $h_4 = \omega_1^1\omega_1^2$; with $\omega_0^1 = 0.5(\cos(x_1)+1)$, $\omega_1^1 = 1-\omega_0^1$, $\omega_0^2 = (\sin(x_1)/x_1 + 0.2167)/1.2167$, and $\omega_1^2 = 1-\omega_0^2$. Applying conditions in Theorem 1 with $\mathrm{V}_0^P = \mathrm{V}_0^G = \varnothing$ and $H_0^P = H_0^G = H_0^L = \mathrm{V}_0^L = \{0\}$ (this configuration corresponds to Theorem 2 in [41]) the following values have been obtained:

$$P_1 = \begin{bmatrix} 0.60 & -0.36 \\ -0.36 & 0.46 \end{bmatrix},\ P_2 = \begin{bmatrix} 0.66 & -0.32 \\ -0.32 & 0.45 \end{bmatrix},\ P_3 = \begin{bmatrix} 0.68 & -0.34 \\ -0.34 & 0.45 \end{bmatrix},\ P_4 = \begin{bmatrix} 0.75 & -0.18 \\ -0.18 & 0.41 \end{bmatrix},\ L_{11} = \begin{bmatrix} 0.06 \\ -0.12 \end{bmatrix},$$

$$L_{22} = \begin{bmatrix} -0.14 \\ -0.14 \end{bmatrix},\ L_{32} = \begin{bmatrix} -0.44 \\ -0.04 \end{bmatrix},\ L_{41} = \begin{bmatrix} -0.57 \\ -0.16 \end{bmatrix},\ G_1 = \begin{bmatrix} 0.38 & -0.13 \\ -0.08 & 0.38 \end{bmatrix},\ \text{and}\ G_2 = \begin{bmatrix} 0.32 & -0.07 \\ -0.13 & 0.37 \end{bmatrix}.$$

Thus, an observer of the form (12) has been designed. Simulation results are shown in Figure 1 for the initial conditions $x(0) = [1 \quad -1]^T$ and $\hat{x}(0) = [0 \quad 0]^T$; the input is $u(t) = 0.5\sin(t)$.

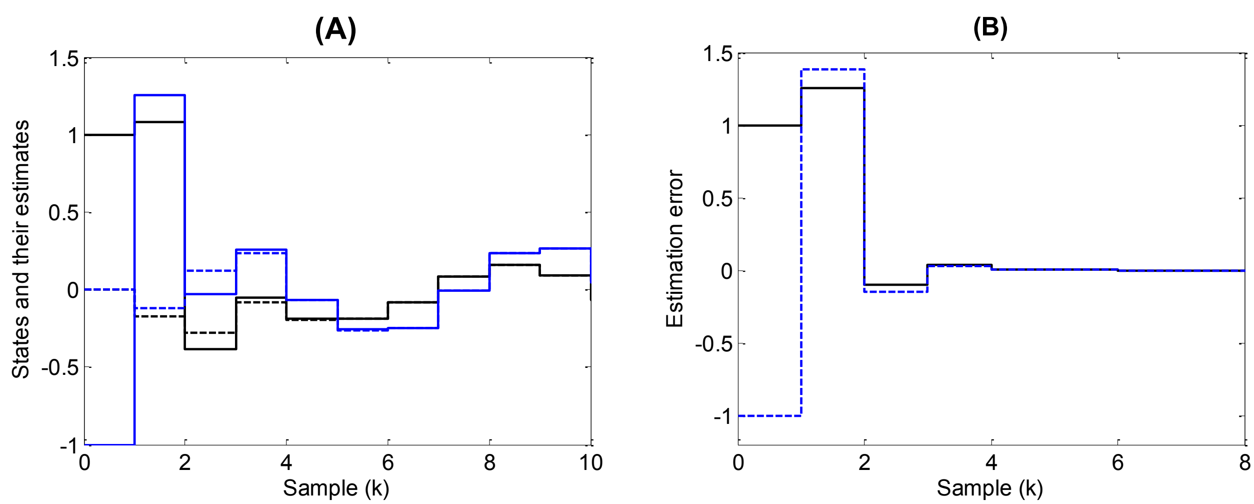

Figure 1. Simulation results: (A) States (black-line for x_1 and blue-line for x_2) and their estimates (dashed lines); (B) The estimation error for Example 1.

Recall that via the approaches given in [9], [15], no solution was found when a standard TS model from the given nonlinear descriptor model. However, via a TS descriptor model and conditions in Theorem 1, a feasible solution is obtained, with only 132 LMI constraints instead of 4112. Table 3 summarizes these findings.

Approach	Number of scalar decision variables	Number of LMIs	Number of rows	Feasible solution
Standard TS + NQ in [9]	144	4112	16416	NO
Standard TS + DNQ in [15]	1584	4112	16416	NO
TS descriptor + Theorem 1	44	132	520	YES

Table 3. Computational complexity indexes for Example 1.

This example clearly illustrates the importance of keeping the descriptor representation. ♦

Example 4. Consider a nonlinear discrete-time descriptor model (1) with $u_k = 0$ and matrices as follows

$$E(x)=\begin{bmatrix} 0.9 & 1+a\left(f_L(x_2)-1\right) \\ -0.4-b\left(f_L(x_2)-1\right) & 1.1 \end{bmatrix},\ A(x)=\begin{bmatrix} -1 & 1+af_R(x_2) \\ -1.5 & 0.5 \end{bmatrix},\ \text{and}\ C(x)=\begin{bmatrix} 0 \\ 1-bf_R(x_2) \end{bmatrix}^T;$$

where the nonlinear functions are $f_R(x_2)=\cos(x_2)$ and $f_L(x_2)=2/\left(1+x_2^2\right)$. Consider the compact set $\Omega_x=\{x\in$ □ . The parameters are defined as $a\in[-1.5,1.5]$ and $b\in[-1.5,1.5]$. In Ω_x, both nonlinear functions are smooth and bounded as follows $f_R(x_2)\in[-1,1]$, $f_L(x_2)\in[0,2]$. This produces a global [3] sector nonlinearity. An equivalent TS descriptor model can be constructed with $r_a = r_e = 2$,

$$E_1=\begin{bmatrix} 0.9 & 0.1+a \\ -0.4-b & 1.1 \end{bmatrix},\ E_2=\begin{bmatrix} 0.9 & 0.1-a \\ -0.4+b & 1.1 \end{bmatrix},\ A_1=\begin{bmatrix} -1 & 1+a \\ -1.5 & 0.5 \end{bmatrix},\ A_2=\begin{bmatrix} -1 & 1-a \\ -1.5 & 0.5 \end{bmatrix}, C_1=\begin{bmatrix} 0 \\ 1-b \end{bmatrix}^T,$$

and $C_2=\begin{bmatrix} 0 \\ 1+b \end{bmatrix}^T$. The MFs are defined as follows $h_1 = 0.5\left(\cos(x_2)+1\right)$, $h_2 = 1-h_1$, $v_1 = 2/\left(1+x_2^2\right)$, and $v_2 = 1-v_1$.

Figure 2 shows the feasible regions for three different configurations of Theorem 1:

- Configuration 1 [41]: Multisets $V_0^P = V_0^G = \varnothing$ and $H_0^P = H_0^G = H_0^L = V_0^L = \{0\}$ (□ .
- Configuration 2: Multisets $H_0^G = H_0^L = V_0^G = \{0\}$, $V_0^L = \{0,0\}$, and $H_0^P = V_0^P = \varnothing$ $(\times)$.
- Configuration 3: Multisets $H_0^G = H_0^L = V_0^G = \{0,-1\}$, $V_0^L = \{0,0,-1\}$, and $H_0^P = V_0^P = \{-1\}$ $(+)$.

Figure 2 shows that conditions in Theorem 1 outperform the ones presented in [41], i.e., Configuration 1. Note that Theorem 1 can be used in several ways, depending on the selection of the multisets as it was discussed before. Selecting multisets as $H_0^G = H_0^L = \mathrm{V}_0^G = \{0\}$, $\mathrm{V}_0^L = \{0,0\}$, and $H_0^P = \mathrm{V}_0^P = \varnothing$, one obtains two double convex sums as (5), therefore Lemma 2 has been implemented. On the other hand choosing $H_0^G = H_0^L = \mathrm{V}_0^G = \{0,-1\}$, $\mathrm{V}_0^L = \{0,0,-1\}$, and $H_0^P = \mathrm{V}_0^P = \{-1\}$ yields conditions of the form (21).

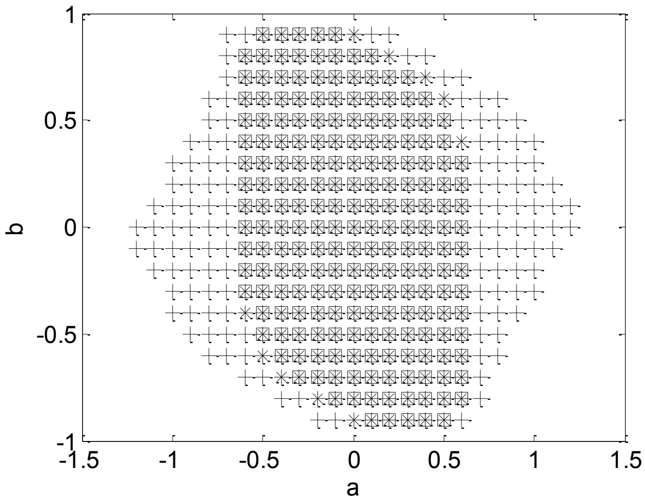

Figure 2. Feasible regions for several values for Example 4.

Configuration 3 performs better than the others due to the number of sums, but it also has a higher computation complexity. ♦

4. Extension to α-sample variation

This section shows an extension of the previous approach via the so-called α-sample variation. This approach has been stated in [11], where the stability analysis of discrete-time standard TS models has been relaxed by replacing the classical one-sample variation of the Lyapunov function $\left(V\left(x(k+1)\right)-V\left(x(k)\right)<0\right)$ by its variation over several samples $\left(V\left(x(k+\alpha)\right)-V\left(x(k)\right)<0\right)$. In other words, instead of asking the variation of the Lyapunov function to decrease at each consecutive sample, it is required to decrease at each α sample; α plays the role of a complexity parameter, as the more α increases, the more complex the conditions (number of variables and of LMI constraints) and the less conservative are the results. By using this idea, the following result can be stated.

Theorem 2: The estimation error dynamics in (13) are asymptotically stable if there exist $P_{\boldsymbol{i}_s^P,\boldsymbol{j}_s^P} = P_{\boldsymbol{i}_s^P,\boldsymbol{j}_s^P}^T$, $\boldsymbol{i}_s^P = pr_{H_s^P}^{\boldsymbol{i}}$, $\boldsymbol{j}_s^P = pr_{\mathrm{V}_s^P}^{\boldsymbol{j}}$, $s = 0,\alpha$, $L_{\boldsymbol{i}_l^L,\boldsymbol{j}_l^L}$, $\boldsymbol{i}_l^L = pr_{H_l^L}^{\boldsymbol{i}}$, $\boldsymbol{j}_l^L = pr_{\mathrm{V}_l^L}^{\boldsymbol{j}}$, and $G_{\boldsymbol{i}_l^G,\boldsymbol{j}_l^G}$, $\boldsymbol{i}_l^G = pr_{H_l^G}^{\boldsymbol{i}}$, $\boldsymbol{j}_l^G = pr_{\mathrm{V}_l^G}^{\boldsymbol{j}}$, $\boldsymbol{i} \in \mathbf{I}_{H_0^\Gamma}$, $\boldsymbol{j} \in \mathbf{I}_{\mathrm{V}_0^\Gamma}$, $l \in \{0,1,\ldots$; where $H_0^\Gamma = H_0^P \cup H_\alpha^P \cup \bigcup$ $\bigcup$, $\mathrm{V}_0^\Gamma = \mathrm{V}_0^P \cup \mathrm{V}_\alpha^P \cup \bigcup$ V $\bigcup$ $(\mathrm{V}$ $\mathrm{V}_l^E)$ such that

$$
\begin{bmatrix}
-P_{H_0^P V_0^P} & (*) & 0 & \cdots & 0 \\
\begin{pmatrix} G_{H_0^G V_0^G} A_{H_0^A} \\ -L_{H_0^L V_0^L} C_{H_0^C} \end{pmatrix} & -G_{H_0^G V_0^G} E_{V_0^E} + (*) & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & {}_{\alpha-2}V_{\alpha-2} \quad V_{\alpha-2}^E + (*) & (*) \\
0 & \cdots & 0 & \begin{pmatrix} G_{H_{\alpha-1}^G V_{\alpha-1}^G} A_{H_{\alpha-1}^A} \\ -L_{H_{\alpha-1}^L V_{\alpha-1}^L} C_{H_{\alpha-1}^C} \end{pmatrix} & \begin{pmatrix} -G_{H_{\alpha-1}^G V_{\alpha-1}^G} E_{V_{\alpha-1}^E} + (*) \\ + P_{H_\alpha^P V_\alpha^P} \end{pmatrix}
\end{bmatrix} < 0. \tag{25}
$$

Proof. Consider the Lyapunov function (14) and its α -sample variation as follows [11], [15], [16]:

$$
\begin{aligned}
\Delta V_\alpha &= V\left(e(k+\alpha)\right) - V\left(e(k)\right) \\
&= e_{k+\alpha}^T P_{H_\alpha^P V_\alpha^P} e_{k+\alpha} - e_k^T P_{H_0^P V_0^P} e_k \\
&= \begin{bmatrix} e_k \\ e_{k+1} \\ \vdots \\ e_{k+\alpha} \end{bmatrix}^T
\begin{bmatrix} -P_{H_0^P V_0^P} & 0 & \cdots & \\ 0 & 0 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {}_{\alpha}V_{\alpha} \end{bmatrix}
\begin{bmatrix} e_k \\ e_{k+1} \\ \vdots \\ e_{k+\alpha} \end{bmatrix} < 0.
\end{aligned} \tag{26}
$$

The error dynamics (13) on α-samples can be summarized in the following equality constraints:

$$
\begin{bmatrix}
S_0 & -E_{V_0^E} & 0 & \cdots & \\
0 & S_1 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & V_{\alpha-2} & 0 \\
0 & \cdots & & S_{\alpha-1} & -E_{V_{\alpha-1}^E}
\end{bmatrix}
\begin{bmatrix} e_k \\ \\ \vdots \\ e_{k+\alpha} \end{bmatrix} = 0. \tag{27}
$$

with $S_l = A_{H_l^A} - G_{H_l^G V_l^G}^{-1} L_{H_l^L V_l^L} C_{H_l^C}$, $l \in \{0,1,\ldots$. Applying Lemma 3, inequality (26) under constraints (27) is equivalent to finding $\mathrm{M} \in$ such that:

$$
\mathrm{M}
\begin{bmatrix}
S_0 & -E_{V_0^E} & 0 & \cdots & \\
0 & S_1 & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & V_{\alpha-2} & 0 \\
0 & \cdots & & S_{\alpha-1} & -E_{V_{\alpha-1}^E}
\end{bmatrix} + (*) +
\begin{bmatrix} -P_{H_0^P V_0^P} & 0 & \cdots & \\ 0 & 0 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & {}_{\alpha}V_{\alpha} \end{bmatrix} < 0. \tag{28}
$$

In order to obtain strict LMI conditions a natural choice [11] of the matrix M is:

$$
M_{H_0^G V_0^G \ldots \; V_{\alpha-1}^G} =
\begin{bmatrix}
0 & 0 & \cdots & \\
G_{H_0^G V_0^G} & 0 & \cdots & \\
0 & G_{H_1^G V_1^G} & \ddots & \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & {}_{\alpha-1}V_{\alpha-1}^G
\end{bmatrix},
$$

leading to (25), thus concluding the proof. ■

Example 5. Consider a discrete TS descriptor model as (2) with $u_k = 0$, $r_a = r_e = 2$, $A_1 = \begin{bmatrix} -1+a & -0.2 \\ -1.5 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 & 2.2 \\ -1.5 & 0.5 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 1.2-b \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 0 \\ 1.2+b \end{bmatrix}^T$, $E_1 = \begin{bmatrix} 0.9 & -1.1 \\ -0.2 & 1.1 \end{bmatrix}$, and $E_2 = \begin{bmatrix} 0.9 & 1.3 \\ -0.6 & 1.1 \end{bmatrix}$. The parameters are defined as $a \in [-0.5, 1]$ and $b \in [-0.5, 0.5]$. Defining multisets of Lyapunov matrix and observer gains matrices as $H_0^G = H_0^L = V_0^G = \{0, -1\}$, $V_0^L = \{0, 0, -1\}$, and $H_0^P = V_0^P = \{-1\}$. Two sets of conditions have been tested. Figure 3 shows the feasible set for:

- Conditions in Theorem 2 for $\alpha = 1$, i.e., the conditions in Theorem 1 $(\square)$.
- Conditions in Theorem 2 for $\alpha = 2$ $(\times)$.

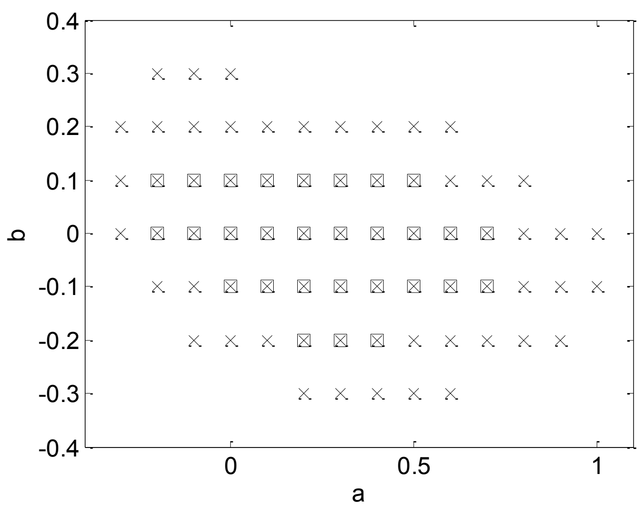

Figure 3. Feasible solution set for Theorem 2 for α=1 (□) and α=2 (×) in Example 5. ♦

5. Concluding remarks

In this paper, novel LMI conditions for the observer design of discrete-time nonlinear descriptor models have been established. The descriptor models under investigation are assumed to have a non-constant full-rank descriptor matrix. Such descriptor systems have been expressed as TS descriptor models. Using delayed non-quadratic Lyapunov functions and non-PDC-like observers, LMI conditions are achieved. An arbitrary number of past samples can be systematically added in order to relax conditions, thus the proposed approach generalizes the previous ones in the literature. The validity of the proposed methodology has been illustrated via numerical examples.

Based on proposed methodology, future directions are numerous. A first one, following [13] or [37] is to derive asymptotically necessary and sufficient conditions. A second one, very challenging, is to consider MFs with unmeasured premise variables such as [42] using properties on $h_i(z(k)) - h_i(\hat{z}(k))$, i.e., Lipschitz or Mean Value Theorem. A third one, a natural extension of observer design, concerns fault detection and can be worked following [43]–[45]. A fourth one is an important issue and corresponds to output feedback control such as [40], [46] and develop observer-based control conditions for TS descriptor models. At last, although conditions may appear with some complexity, their use to real-time applications does not present problems and are already used for biomechanical applications such as [8], [20], or very recently for seated disabled people (work in progress).

6. Acknowledgments

This work is supported by the Ministry of Higher Education and Research, the CNRS, the Nord-Pas-de-Calais Region (grant No. 2013_11067), and a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS – UEFISCDI, project number PN-II-RU-TE-2014-4-0942, contract number 88/01.2015. The authors gratefully acknowledge the support of these institutions.

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Appendix A

Proof of Lemma 2. Applying Lemma 2 on the double convex sum of $h(z)$ in (5) produces

$$\begin{aligned}&\Upsilon_{i_1 i_1}^{vv} < 0, \quad \forall i_1,\\ &\frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\},\ i_1 \neq i_2.\end{aligned} \tag{29}$$

Additionally, using Lemma 2 for the first inequality of (29) gives

$$\Upsilon_{i_1 i_1}^{vv} < 0 \Leftarrow \begin{cases}\Upsilon_{i_1 i_1}^{j_1 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, j_1 \in \{1,2,\ldots,r_e\},\\ \dfrac{2}{r_e - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1} < 0, \quad \forall i_1 \in \{1,2,\ldots,r_a\}, j_1, j_2 \in \{1,2,\ldots,r_e\}, \quad j_1 \neq j_2.\end{cases} \tag{30}$$

Following a similar procedure with the second inequality in (29), we obtain:

$$\begin{aligned}&\frac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{vv} + \Upsilon_{i_1 i_2}^{vv} + \Upsilon_{i_2 i_1}^{vv} < 0\\ &\Leftarrow \begin{cases}\dfrac{2}{r_a - 1}\Upsilon_{i_1 i_1}^{j_1 j_1} + \Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\}, j_1 \in \{1,2,\ldots,r_e\}, \quad i_1 \neq i_2,\\ \dfrac{4}{(r_e - 1)(r_a - 1)}\Upsilon_{i_1 i_1}^{j_1 j_1} + \dfrac{2}{r_e - 1}\left(\Upsilon_{i_1 i_2}^{j_1 j_1} + \Upsilon_{i_2 i_1}^{j_1 j_1}\right) + \dfrac{2}{r_a - 1}\left(\Upsilon_{i_1 i_1}^{j_1 j_2} + \Upsilon_{i_1 i_1}^{j_2 j_1}\right) + \Upsilon_{i_1 i_2}^{j_1 j_2} + \Upsilon_{i_2 i_1}^{j_1 j_2}\\ \quad + \Upsilon_{i_1 i_2}^{j_2 j_1} + \Upsilon_{i_2 i_1}^{j_2 j_1} < 0, \quad \forall i_1, i_2 \in \{1,2,\ldots,r_a\}, j_2, j_1 \in \{1,2,\ldots,r_e\},\ i_1 \neq i_2, \quad j_1 \neq j_2.\end{cases}\end{aligned} \tag{31}$$

This concludes the proof. ■