# Local observer design for discrete-time TS systems ⋆

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**Abstract:** In this paper, the problem of local observer design for discrete-time Takagi-Sugeno fuzzy systems is addressed. The observers being designed are local in the sense that relaxed asymptotic convergence conditions are sought locally, rather than globally, while also determining the region where the conditions are valid. The design conditions are derived using both quadratic and non-quadratic Lyapunov functions. The obtained results are expressed in terms of linear matrix inequalities and they are illustrated on a numerical example.

*Keywords:* observer design, Takagi-Sugeno fuzzy systems, local stability, LMI conditions

## 1. INTRODUCTION

Since the work of Takagi and Sugeno (1985), a lot of attention has been paid to nonlinear systems represented by the so-called Takagi-Sugeno (TS) models. The TS models consist in a time-varying interpolation of several linear models. The main appealing property of the obtained global system is its ability to exactly describe the original nonlinear system, at least in a compact set of the state and input space (Tanaka and Wang, 2001). Thanks to the linear structure of the submodels many tools have been adapted from linear control theory, among then the powerful and tractable linear matrix inequality (LMI)-based (Boyd et al., 1994) analysis and synthesis.

Starting from the seminal works of (Tanaka and Sano, 1994; Tanaka et al., 1998; Wang et al., 1996), the LMI-based stability analysis and the derived controller and observer designs received a considerable amount of attention. Most of the obtained results are derived from Lyapunov analysis and more specifically based on the use of different types of Lyapunov function candidates. The first and simplest one to be used was the quadratic Lyapunov function (Wang et al., 1996; Tanaka et al., 1998; Tanaka and Wang, 2001; Tuan et al., 2001). The obtained results are known to be conservative since a common Lyapunov matrix is sought to satisfy the set of LMI constraints. Fuzzy or multiple Lyapunov functions were introduced to bring more degrees of freedom in the LMI optimization (Tanaka et al., 2003; Mozelli et al., 2009). More recently non-quadratic Lyapunov functions (roughly defined with the inverse of the fuzzy Lyapunov gain) were proposed (Feng, 2006; Tanaka et al., 2007; Guerra et al., 2012). Checking stability generally reduces to checking the negativity of multiple polytopic sums. By the appropriate factorizations of these sums the conditions can be relaxed, leading eventually to asymptotically necessary and sufficient (ANS) conditions in (Sala and Ariño, 2007). The ANS conditions go along with an increasing number of LMI to be satisfied and consequently to numerical intractability. Other ways to relax the LMI optimization process involve the introduction of slack variables (Mozelli et al., 2009) or using the so-called descriptor redundancy (Tanaka et al., 2007; Guelton et al., 2009) approach.

In the continuous-time case, the main inconvenient in the fuzzy Lyapunov function approach is that differentiating the Lyapunov function means differentiating the membership functions. Deriving tractable conditions while linking these derivatives to the system states is challenging. In (Tanaka et al., 2003), the proposed solution is to assume known upper bounds on the time derivative of the membership functions. First, this assumption induces local results: they are valid on the validity domain of the bounding assumption and a key point is then to enlarge this domain (Lee and Kim, 2014). Second, these upper bounds are far from being trivial to compute.

In the discrete time case, solutions are generally global. Since the time derivative of the Lyapunov candidate is no longer needed, non-quadratic Lyapunov functions have shown a real improvement (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011) for developing global stability and design conditions. This fact also allows some relaxations by calculating this difference between $\alpha$ – instead of the usual two – consecutive instants (Kruszewski et al., 2008) or by considering delayed Lyapunov functions (Lendek et al.,

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2015). Since no further conditions are introduced, if the developed conditions are feasible, then stability of the corresponding system is ensured globally – actually in the largest Lyapunov level set included in the domain where the TS model is defined. However, it is possible that stability (of the closed-loop system or error dynamics, as might be the case) cannot be ensured on the full domain where the TS model is defined, but by reducing this domain, the conditions become feasible (Pitarch et al., 2011).

Based on the above considerations, and recalling that TS systems are in effect nonlinear systems, in this paper we consider the design of local TS observers. At the same time, we aim to find the region where the estimation error can be proven stable. The observer design mainly uses the theoretical tools introduced in (Lendek and Lauber, 2016) concerning the local stability.

The remainder of the paper is organized as follows. Section 2 is devoted to the problem formulation and some backgrounds and notations that are used afterwards. The main results are presented in Section 3 and are illustrated on a numerical example in Section 4. Finally, in Section 5 some concluding remarks are given.

## 2. PROBLEM FORMULATION AND PRELIMINARIES

The considered discrete-time nonlinear systems are defined as

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= g(x(k), u(k)) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{n_u}$, $y(k) \in \mathbb{R}^{n_y}$ are respectively the state vector, the control input, the measured output. It is known (Tanaka and Wang, 2001) that, at least in a compact set $\mathcal{D}$ of the input and state (1) can be described, without any approximation, by the discrete time TS model defined by

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} h_i(z(k))(A_i x(k) + B_i u(k)) \\ y(k) &= \sum_{i=1}^{r} h_i(z(k))(C_i x(k) + D_i u(k)) \end{aligned} \tag{2}$$

where the functions $h_i$ are the membership functions depending on the decision variable $z(k) \in \mathbb{R}^q$ involving the input, the output and/or the system state. Each membership function $h_i(.)$ quantifies the relative importance of the $i^{th}$ submodel $(A_i, B_i, C_i, D_i)$ in the global nonlinear system (2). These functions satisfy the convex sum properties

$$\sum_{i=1}^{r} h_i(z(k)) = 1 \quad \text{and} \quad 0 \leq h_i(z(k)) \leq 1, \quad \forall k \geq 0 \tag{3}$$

**Notations.** The notation $*$ is used for the blocks induced by symmetry, for any square matrix $M$, $\mathbb{S}(M)$ is defined by $\mathbb{S}(M) = M + M^T$, $I$ is the identity matrix, 0 is the null matrix of appropriate dimensions. A block diagonal matrix with diagonal blocks $P_i$ is denoted $diag(P_1, P_2, \dots)$. The set of the $r$ first strictly positive integers is denoted by $\mathcal{I}_r = \{1, 2, \dots, r\}$. For any sets of matrices $X_i$, with $i \in \mathcal{I}_r$, the polytopic matrices $X_h$, $X_h^{-1}$, and $X_{h+}$ are defined by

$$X_h = \sum_{i=1}^{r} h_i(z(k)) X_i, \quad X_{h+} = \sum_{i=1}^{r} h_i(z(k+1)) X_i$$
$$X_h^{-1} = \left( \sum_{i=1}^{r} h_i(z(k)) X_i \right)^{-1}$$

The polytopic matrices obtained from a double (triple) summation of $X_{ij}$, for $(i,j) \in \mathcal{I}_r^2$ ($X_{ijk}$, for $(i,j,k) \in \mathcal{I}_r^3$) are denoted by $X_{hh}$ ($X_{hhh+}$) and defined as

$$\begin{aligned} X_{hh} &= \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z(k)) h_j(z(k)) X_{ij} \\ X_{hhh+} &= \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} h_i(z(k)) h_j(z(k)) h_k(z(k+1)) X_{ijk} \end{aligned} \tag{4}$$

In what follows, the decision variable $z(k)$ is assumed to be real time accessible (e.g. known exogenous signal, input signal).

Our objective is to design a state observer of the form

$$\begin{aligned} \hat{x}(k+1) &= A_h \hat{x}(k) + B_h u(k) + M_h^{-1} L_h (y(k) - \hat{y}(k)) \\ \hat{y}(k) &= C_h \hat{x}(k) + D_h u(k) \end{aligned} \tag{5}$$

The goal reduces to find the observer gains $M_h$ and $L_h$ such that the state estimate $\hat{x}$ asymptotically converges towards the system state $x$. In order to relax the conditions, we search for a domain $\mathcal{D}_R$, defined as the domain where the following inequality in the state estimation error $e(k) = x(k) - \hat{x}(k)$ is satisfied for some symmetric matrix $R$

$$\begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix}^T R \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} > 0 \tag{6}$$

Obviously the obtained results are only valid in a specific domain of the state and error space, meaning that the asymptotical convergence of the state estimation error to zero is not guaranteed for any possible trajectories in the whole definition domain of (2).

**Remark:** Note that for any $R > 0$, (6) will hold. However, due to the error dynamics, the actual domain where the error may be situated may be reduced to $\emptyset$.

The derived results will be obtained by exploiting two kinds of Lyapunov functions depending on the state estimation error. The first one is a quadratic Lyapunov function and the second one is a non-quadratic one. They are defined by

$$V(e(k)) = e^T(k) P e(k) \tag{7}$$

where $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite and by

$$\tilde{V}(e(k)) = e^T(k) P_h e(k) \tag{8}$$

where $P_i \in \mathbb{R}^{n \times n}$ for $i \in \mathcal{I}_r$ are symmetric positive definite matrices.

In what follows, we will use the following lemmas. The two first ones are the well known S-procedure and the Finsler lemma and the two other ones are classical relaxation schemes used to check the negativity of a double sum $X_{hh}$.

*Lemma 1.* (S-procedure). (Boyd et al., 1994) Consider some matrices $F_i = F_i^T \in \mathbb{R}^{\ell \times \ell}$ and a vector $\boldsymbol{z} \in \mathbb{R}^{\ell}$, such that $\boldsymbol{z}^T F_i \boldsymbol{z} \geq 0$, $i \in \mathcal{I}_p$. If there exist $\tau_i \geq 0$, $i \in \mathcal{I}_p$, such that

$$F_0 - \sum_{i=1}^{p} \tau_i F_i > 0$$

then the following quadratic inequality holds for $\boldsymbol{z} \neq 0$.

$$\boldsymbol{z}^T F_0 \boldsymbol{z} > 0 \quad (9)$$

*Lemma 2.* (Finsler lemma). (Skelton et al., 1998) For any vector $\boldsymbol{z} \in \mathbb{R}^\ell$ and any matrices $Q = Q^T \in \mathbb{R}^{\ell\times\ell}$ and $S \in \mathbb{R}^{m\times\ell}$ such that $rank(S) < \ell$, the two following statements are equivalent.

(1) $\boldsymbol{z}^T Q \boldsymbol{z} < 0,\ \boldsymbol{z} \in \mathbb{R}^\ell$, such that $\boldsymbol{z} \neq 0$ and $S\boldsymbol{z} = 0$
(2) $\exists \mathcal{M} \in \mathbb{R}^{\ell\times m}$ such that $Q + \mathbb{S}(\mathcal{M}S) < 0$

*Lemma 3.* (Wang et al., 1996) The inequality

$$x^T X_{hh} x < 0$$

holds $\forall x \in R^n$ if the following inequalities are satisfied

$$0 > X_{ii},\ i \in \mathcal{I}_r \quad (10a)$$

$$0 > X_{ij} + X_{ji},\ (i,j) \in \mathcal{I}_r^2,\ i < j \quad (10b)$$

*Lemma 4.* (Tuan et al., 2001) The inequality

$$x^T(k) X_{hh} x(k) < 0$$

holds $\forall x \in R^n$ if the following inequalities are satisfied

$$0 > X_{ii},\ i \in \mathcal{I}_r \quad (11a)$$

$$0 > \frac{2}{r-1} X_{ii} + X_{ij} + X_{ji},\ (i,j) \in \mathcal{I}_r^2,\ i \neq j \quad (11b)$$

## 3. LOCAL OBSERVER DESIGN FOR TS SYSTEMS

This section presents our main results. The observer design aims at finding the gains $M_h$ and $L_h$ guaranteeing a local convergence of the state estimation error. Under the assumption that the scheduling vector depends only on measured variables, the estimation error dynamics, from (2) and (5), are given by

$$e(k+1) = (A_h - M_h^{-1} L_h C_h) e(k) \quad (12)$$

First, the proposed observer (5) is designed using a common quadratic Lyapunov function, after which the local non-quadratic stabilization of the estimation error will be studied.

### *3.1 Quadratic design*

The following theorem details the existence condition of the observer (5) in the quadratic case.

*Theorem 5.* There exists an observer of the form (5) for the system (2), such that the state estimation error $e(k)$ asymptotically converges towards the origin in the largest Lyapunov level set included in the domain $\mathcal{D} \cap \mathcal{D}_R$, if there exists a positive scalar $\varepsilon$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a symmetric matrix $R \in \mathbb{R}^{2n\times 2n}$, matrices $M_i \in \mathbb{R}^{n\times n}$ and $L_i \in \mathbb{R}^{n\times n_y}$, for $i \in \mathcal{I}_r$, satisfying the following matrix inequality

$$\begin{pmatrix} \mathbb{S}(\varepsilon(M_h A_h - L_h C_h)) - P & \star \\ M_h A_h - L_h C_h - \varepsilon M_h^T & P - \mathbb{S}(M_h) \end{pmatrix} + R < 0 \quad (13)$$

**Proof.** In order to prove the local asymptotic convergence of the state estimation error, we have to prove the negativeness of the difference of the quadratic Lyapunov function, namely

$$\Delta V(k) = \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} < 0 \quad (14)$$

in $\mathcal{D}_R$ defined by (6) and along the trajectory of (12). The dynamics of the state estimation error (12) can also be written as the following equality constraint

$$\begin{pmatrix} A_h - M_h^{-1} L_h C_h & -I \end{pmatrix} \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} = 0 \quad (15)$$

Defining $\mathcal{M}_h$ by

$$\mathcal{M}_h = \begin{pmatrix} \varepsilon M_h \\ M_h \end{pmatrix} \quad (16)$$

the inequality (13) can be written as

$$R + \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} + \mathbb{S}\left(\mathcal{M}_h \begin{pmatrix} A_h - M_h^{-1} L_h C_h & -I \end{pmatrix}\right) < 0 \quad (17)$$

Pre- and post-multiplying (17) by $z^T = (e^T(k)\ e^T(k+1))$ and $z$, from Lemma 2, it implies that

$$\begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix}^T \left( \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix} - R \right) \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} > 0 \quad (18)$$

holds along the trajectory of (12). From Lemma 1, with $\tau = 1$, $F_1 = R$ and $F_0 = diag(P, -P)$, inequality (18) implies that, in $\mathcal{D}_R$ (i.e. if $z^T R z > 0$ is satisfied), we have

$$z^T diag(P, -P) z > 0 \quad (19)$$

This last inequality is equivalent to (14) and achieves the proof.

Since the matrix inequality (13) is based on a double summation both involving the same weighting functions $h_i(z(k))$, the relaxation schemes in Lemmas 3 or 4 can easily be used to derive sufficient conditions that are numerically tractable with classical dedicated softwares. Moreover, in this case, the observer gains in (5) are directly obtained from the numerical solution of the LMI problem given in the following corollary.

*Corollary 6.* There exists an observer (5) for the system (2), such that the state estimation error $e(k)$ asymptotically converges towards the origin in the largest Lyapunov level set included in $\mathcal{D} \cap \mathcal{D}_R$, if there exists a positive scalar $\varepsilon$, a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a symmetric matrix $R \in \mathbb{R}^{2n\times 2n}$, matrices $M_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I}_r$ and $L_i \in \mathbb{R}^{n\times n_y}$ satisfying (10) or (11) where $X_{ij}$, for $(i,j) \in \mathcal{I}_r^2$, are defined as

$$X_{ij} = \begin{pmatrix} \mathbb{S}(\varepsilon(M_j A_i - L_i C_j)) - P & \star \\ M_j A_i - L_i C_j - \varepsilon M_j^T & P - \mathbb{S}(M_j) \end{pmatrix} + R \quad (20)$$

**Remark:** Due to $\varepsilon$, the conditions in Corollary 6 are parameterized LMIs. To obtain LMIs, $\varepsilon$ may be fixed or can be searched for in a logarithmically spaced family of values (Oliveira and Peres, 2007), $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$.

### *3.2 Non-quadratic design*

As widely known, the use of a common quadratic Lyapunov function may introduce some conservatism. In order to obtain a more general result, the non quadratic Lyapunov function (8) is exploited here and the following theorem details the existence condition of the observer (5) in the non-quadratic case.

*Theorem 7.* There exists an observer (5) for the system (2), such that the state estimation error $e(k)$ asymptotically converges towards the origin in the largest Lyapunov level set included in $\mathcal{D} \cap \mathcal{D}_R$, if there exists a positive scalar $\varepsilon$, symmetric positive definite matrices $P_i \in \mathbb{R}^{n\times n}$, a symmetric matrix $R \in \mathbb{R}^{2n\times 2n}$, matrices $M_i \in \mathbb{R}^{n\times n}$ and $L_i \in \mathbb{R}^{n\times n_y}$, for $i \in \mathcal{I}_r$, satisfying the following inequalities.

$$\begin{pmatrix} \mathbb{S}(\varepsilon(M_hA_h - L_hC_h)) - P_h & \star \\ M_hA_h - L_hC_h - \varepsilon M_h^T & P_{h+} - \mathbb{S}(M_h) \end{pmatrix} + R < 0 \tag{21}$$

**Proof.** The proof follows the same line as the one of the theorem 5, except that the difference of the Lyapunov function (8) is given by

$$\Delta\tilde{V}(k) = \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix}^T \begin{pmatrix} -P_h & 0 \\ 0 & P_{h+} \end{pmatrix} \begin{pmatrix} e(k) \\ e(k+1) \end{pmatrix} \tag{22}$$

Because of the chosen Lyapunov function, the matrix inequality (21) needs a triple summation to be written as an LMI optimization problem and Lemmas 3 and 4 need to be slightly adapted as follows:

*Lemma 8.* The inequality $x^T X_{hhh+} x < 0$ holds $\forall x \in \mathbb{R}^n$ if the following inequalities are satisfied

$$0 > X_{iik},\ (i,k) \in \mathcal{I}_r^2 \tag{23a}$$

$$0 > X_{ijk} + X_{jik},\ (i,j,k) \in \mathcal{I}_r^3,\ i < j \tag{23b}$$

*Lemma 9.* The inequality $x^T X_{hhh+} x < 0$ holds $\forall x \in \mathbb{R}^n$ if the following inequalities are satisfied

$$0 > X_{iik},\ (i,k) \in \mathcal{I}_r^2 \tag{24a}$$

$$0 > \frac{2}{r_1} X_{iik} + X_{ijk} + X_{jik},\ (i,j,k) \in \mathcal{I}_r^3,\ i \neq j \tag{24b}$$

Then the conditions of Theorem 7 can be set up as the parameterized LMI problems given in the following corollary.

*Corollary 10.* There exists an observer (5) for the system (2), such that the state estimation error $e(k)$ asymptotically converges towards the origin in the largest Lyapunov level set included in $\mathcal{D} \cap \mathcal{D}_R$, if there exists a positive scalar $\varepsilon$, a symmetric positive definite matrix $P_i \in \mathbb{R}^{n\times n}$, a symmetric matrix $R \in \mathbb{R}^{2n\times 2n}$, matrices $M_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I}_r$ and $L_i \in \mathbb{R}^{n\times n_y}$ satisfying (23) or (24) where $X_{ijk}$, for $(i,j,k) \in \mathcal{I}_r^3$, is defined as

$$X_{ijk} = \begin{pmatrix} \mathbb{S}(\varepsilon(M_jA_i - L_iC_j)) - P_j & \star \\ M_jA_i - L_iC_j - \varepsilon M_j^T & P_k - \mathbb{S}(M_j) \end{pmatrix} + R \tag{25}$$

Note that, contrary to existing results, for this design it is not necessary that $M_h + M_h^T > 0$. This is because the negativity of the difference in the Lyapunov function can be compensated by the matrix $R$.

**Remark:** for different $\varepsilon$, different $R$ matrices, but at the same time, different observer gains can be obtained. Since the error dynamics will be different for each case, comparing the corresponding domains is far from trivial.

## 4. NUMERICAL EXAMPLE

In what follows, we illustrate the developed conditions on a numerical example.

*Example 11.* Consider the four-rule TS model defined on the domain $\mathcal{D} = \{x_1, x_2 \in [-2, 2]\}$, with local matrices

$$A_1 = \begin{pmatrix} -2 & -1 \\ 1 & 4.25 \end{pmatrix} \qquad C_1 = (1\ 0)$$

$$A_2 = \begin{pmatrix} -2 & 1 \\ 1 & -4.25 \end{pmatrix} \qquad C_2 = (-5\ 0)$$

$$A_3 = \begin{pmatrix} 2 & -1 \\ 1 & 4.25 \end{pmatrix} \qquad C_3 = (1\ 0)$$

$$A_4 = \begin{pmatrix} 2 & 1 \\ 1 & -4.25 \end{pmatrix} \qquad C_4 = (-5\ 0)$$

and membership functions

$$h_1(x_1) = \frac{(2-x_1)}{4}\frac{(1-\sin(x_1))}{2}$$

$$h_2(x_1) = \frac{(2-x_1)}{4}\frac{(1+\sin(x_1))}{2}$$

$$h_3(x_1) = \frac{(2+x_1)}{4}\frac{(1-\sin(x_1))}{2}$$

$$h_4(x_1) = \frac{(2+x_1)}{4}\frac{(1+\sin(x_1))}{2}$$

The above membership functions have the convex sum property.

Note that for this specific TS model, neither a PDC nor a non-PDC observer can be designed, independent of the Lyapunov function used (common quadratic or nonquadratic) – the LMI conditions are unfeasible. In what follows, we study if a local observer can be designed using the conditions of Theorem 5. In order to maximize the domain $\mathcal{D}_R$, the matrix $R$ has been chosen with the following structure

$$R = diag(\bar{R}, -I) \tag{26}$$

where $\bar{R}$ is a decision variable whose trace is being maximized.

The conditions of Theorem 5 with $\varepsilon = 0$ and using Lemma 3 for relaxation are feasible and the following results have been obtained:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$M_1 = 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad L_1 = 10^{-3}\begin{pmatrix} 0.1351 \\ 0.3273 \end{pmatrix}$$

$$M_2 = 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad L_2 = 10^{-3}\begin{pmatrix} 0.0027 \\ -0.2136 \end{pmatrix}$$

$$M_3 = 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad L_3 = 10^{-4}\begin{pmatrix} 0.1377 \\ -0.6871 \end{pmatrix}$$

$$M_4 = 10^{-3}\begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix} \qquad L_4 = 10^{-3}\begin{pmatrix} 0.0316 \\ -0.1324 \end{pmatrix}$$

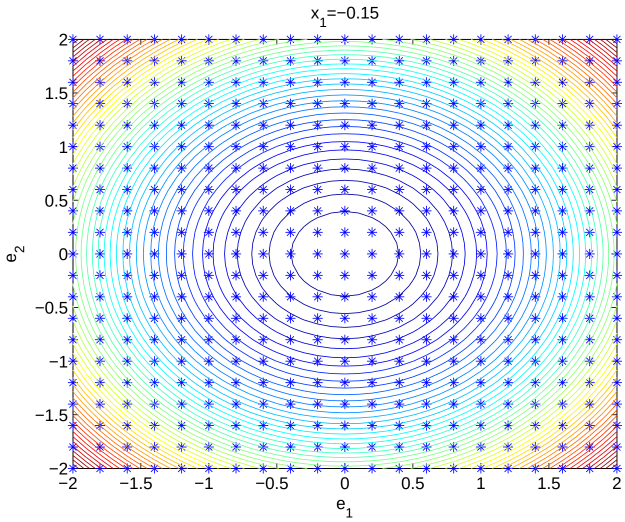
Even if the Lyapunov function in this case is quadratic and thus does not depend on the scheduling variables, one should note that due to its definition, $\mathcal{D}_R$ depends both on $e(k)$ and $e(k+1)$ and thus indirectly depends on the scheduling variable $z(k)$. Consequently, the actual region of attraction $\mathcal{D}_R$ depends on $z(k)$.

To verify in which domain the estimation error converges to zero, the domain $\mathcal{D}_R$ and the Lyapunov level sets are presented in Figure 11. Note that for $x_1 \leq -0.2$ and $x_1 \geq 0.25$, the Lyapunov level sets included in $\mathcal{D}_R$ are reduced to zero, as seen in Figure 2. A trajectory of the error dynamics, with $x(0) = [-0.15,\ -1]^T$ and $e(0) = [-1,\ 1]^T$ is presented in Figure 3.

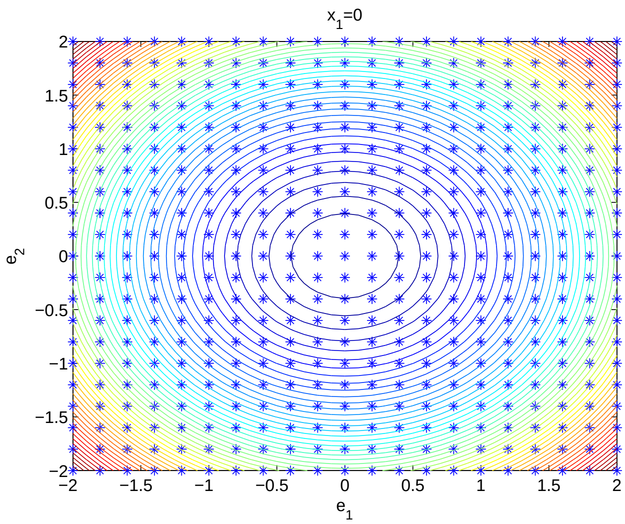
It has to be noted that although the conditions of Theorems 5 and 7 may be used to design local observers, the shortcoming of this design is that the domain has to be verified a posteriori.

## 5. CONCLUSION

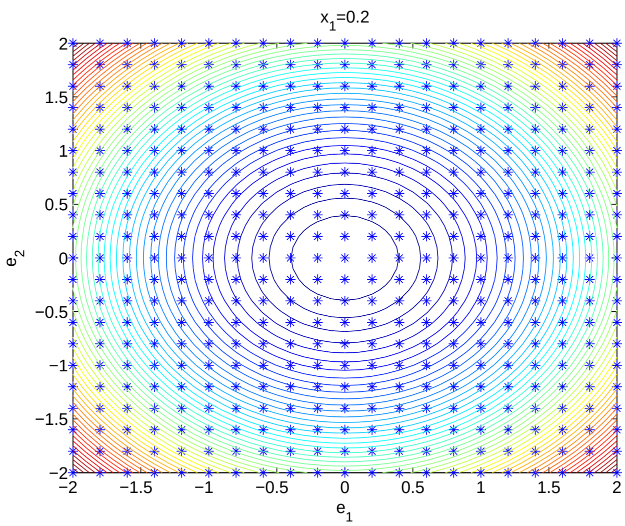
In this paper, LMI conditions have been proposed for the design of local state observers for TS fuzzy systems. Both quadratic and non-quadratic Lyapunov functions have been employed to derive the conditions. In our future work, we will extend the results for $\alpha$ sample variation of the Lyapunov function. The possible improvement obtained with the introduction of delayed Lyapunov functions should also be studied. Finaly, in order to make the verification of the domain of convergence easier, we will use a domain description depending on the measurement error instead of the estimation error.




(a) $x_1 = -0.15$




(b) $x_1 = 0$




(c) $x_1 = 0.2$

Fig. 1. The domain $\mathcal{D}_R$ (*) and the Lyapunov level sets for different values of the scheduling variables.

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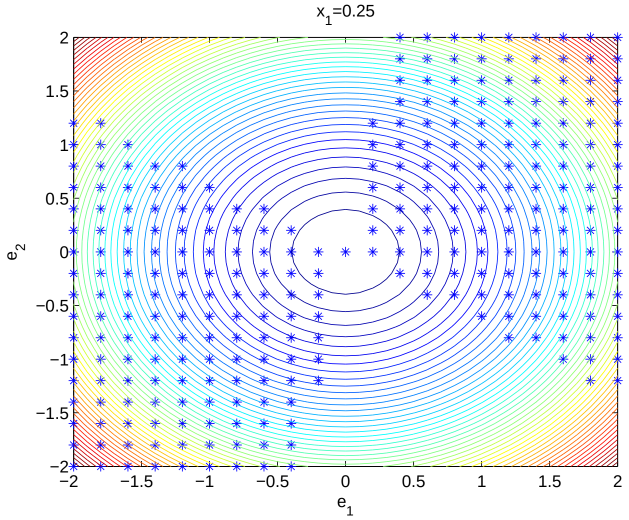


Fig. 2. The domain $\mathcal{D}_R$ (*) and the Lyapunov level sets for $x_1 = 0.25$.

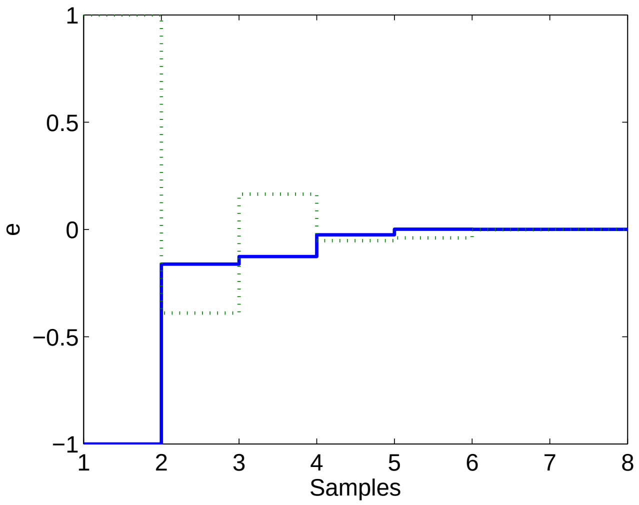


Fig. 3. A trajectory of the estimation error.

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