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Stability analysis and observer design for string-connected TS systems ⋆

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Abstract: Distributed systems consist of interconnected, lower-dimensional subsystems. For such systems, distributed analysis and design present several advantages, such as modularity, easier analysis, and reduced computational complexity. Applications include distributed process control, traffic and communication networks, irrigation systems, hydropower valleys, etc. A special case of distributed systems is when the subsystems are connected in a string. By exploiting such a structure, in this paper, we propose conditions for the distributed stability analysis of Takagi-Sugeno fuzzy systems connected in a string. These conditions are extended to observer design. Sufficient LMI conditions, which are easy to solve are also provided. The approach is illustrated on a simulation example.

Keywords: Stability analysis, state observers, distributed models, fuzzy systems.

1. INTRODUCTION

Takagi-Sugeno (TS) fuzzy systems (Takagi and Sugeno, 1985) are nonlinear, convex combinations of local linear models, and have the property that they are capable to approximate a large class of nonlinear systems to an arbitrary degree of accuracy (Fantuzzi and Rovatti, 1996). For a TS fuzzy model, well-established methods and algorithms exist to analyze its stability or to design observers for it. Several types of observers have been developed for continuous-time TS fuzzy systems, among which: fuzzy Thau-Luenberger observers (Tanaka and Wang, 1997; Tanaka et al., 1998), reduced-order observers (Bergsten et al., 2001, 2002), and sliding-mode observers (Palm and Bergsten, 2000). Most of the stability and design conditions rely on the feasibility of an associated system of linear matrix inequalities (LMIs).

Many physical systems, such as power systems, communication networks, economic systems, and traffic networks are composed of interconnections of lower-dimensional subsystems. Recently, decentralized analysis and control design for such systems has received much attention (Haijun et al., 2006; Liu and Zhang, 2005; Krishnamurthy and Khorrami, 2003; Bavafa-Toosi et al., 2006; Zhang et al., 2006; Liu et al., 2007).

Stability analysis of distributed TS systems mainly relies on the existence of a common quadratic Lyapunov function for each subsystem. Most results make use of the assumption that the number of subsystems and some bounds on the interconnection terms are known a priori, and the analysis of the subsystems is performed in parallel. For instance, an early result that relies on the existence of an M-matrix [1] or positive definite matrices has been formulated by Akar and Özgüner (2000) and Wang and Lin (2005). In these approaches, LMI conditions for establishing the stability of the individual subsystems are solved in parallel, and afterward the stability of the whole system is verified. For hybrid linear-fuzzy systems, a method for establishing the stability of the distributed system has been proposed by Xu et al. (2006). For distributed TS systems with affine consequents, but linear interconnection terms among the subsystems, an approach based on piecewise Lyapunov functions has been developed by Zhang et al. (2006). Stability analysis of uncertain distributed TS systems has been investigated e.g., by Liu and Zhang (2005).

All the above mentioned results assume that any two subsystems in the distributed system may be interconnected. While this assumption makes the results generally applicable, it also introduces conservativeness. In this paper we develop conditions for the stability analysis of string-connected TS systems, i.e., a distributed TS system in which each subsystem is connected only to its two neighbors. System which have such interconnections are e.g., material flow processes, hydropower valleys, irrigation systems. The coupling between the subsystems is realized

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[1] A square matrix M is an M-matrix if the off-diagonal elements are all negative and all the eigenvalues of M have non-negative real part.

through their states, and each subsystem is assumed to be stable. The approach is also extended to observer design, under the assumption that the estimated states are communicated between the neighbors.

The structure of the paper is as follows. Section 2 presents the general form of the TS models and the notations used in this paper. Section 3 describes the proposed stability conditions and Section 4 presents the conditions for observer design. The observer design is illustrated on an example in Section 5. Finally, Section 6 concludes the paper.

2. PRELIMINARIES

In this paper, we consider the following autonomous TS fuzzy system

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i \boldsymbol{x}$$

for stability analysis and the system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i \boldsymbol{x} + B_i \boldsymbol{u})\\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(C_i \boldsymbol{x})\end{aligned}$$

for observer design, where $\boldsymbol{x}$ is the vector of the state variables, $\boldsymbol{u}$ is the input vector, $\boldsymbol{y}$ is the measurement vector. In the equations above, A_i, B_i, and C_i, $i = 1, 2, \ldots, m$ represent the matrices of the ith local linear model and w_i, $i = 1, 2, \ldots, m$ are the corresponding membership functions, which depend on the scheduling variables $\boldsymbol{z}$. The scheduling variables in general may depend on the states, inputs, or other exogenous variables.

Throughout the paper it is assumed that the membership functions are normalized, i.e., $w_i(\boldsymbol{z}) \geq 0$, $\sum_{i=1}^{m} w_i(\boldsymbol{z}) = 1$, $\forall \boldsymbol{z}$. The matrices I and 0, respectively, denote the identity and the zero matrices of the appropriate dimensions, and $\mathcal{H}(A)$ represents the Hermitian of the matrix A, i.e., $\mathcal{H}(A) = A + A^T$.

3. STABILITY ANALYSIS

In this paper we focus on distributed systems where the subsystems are connected in a bidirectional string, as shown in Figure 1.

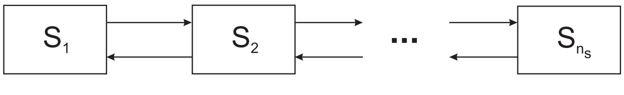

Fig. 1. Subsystems connected in a string.

Such interconnections are common for instance in flow processes or production processes. Each subsystem l, $l = 1, \ldots, n_s$ is described by the TS model

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m_1} w_i^1(\boldsymbol{z}_1)(A_i^1 \boldsymbol{x}_1 + A_i^{1,2}\boldsymbol{x}_2)\\ \dot{\boldsymbol{x}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l \boldsymbol{x}_l + A_i^{l,l-1}\boldsymbol{x}_{l-1} + A_i^{l,l+1}\boldsymbol{x}_{l+1})\\ &\quad l = 2, 3, \ldots, n_s - 1\\ \dot{\boldsymbol{x}}_{n_s} &= \sum_{i=1}^{m_{n_s}} w_i^{n_s}(\boldsymbol{z}_{n_s})(A_i^{n_s}\boldsymbol{x}_{n_s} + A_i^{n_s,n_s-1}\boldsymbol{x}_{n_s-1})\end{aligned} \tag{1}$$

It is assumed that the individual subsystems, i.e., the systems described by $\dot{\boldsymbol{x}}_l = \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l \boldsymbol{x}_l)$, $l = 1, \ldots, n_s$, are globally asymptotically stable, provable by a common quadratic Lyapunov function. Consequently, there exist $P_l = P_l^T > 0$, $l = 1, \ldots, n_s$, so that [2]

$$\mathcal{H}(P_l A_z^l) < 0 \tag{2}$$

where $A_z^l = \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l) A_i^l$. For stability analysis we assume that the scheduling variables $\boldsymbol{z}_l$, $l = 1, \ldots, n_s$, may depend on the states, inputs, and outputs of any of the subsystems or on other exogenous variables.

With the considerations above, the following result can be formulated:

Theorem 1. The distributed TS fuzzy system with the subsystems described by (1) is globally asymptotically stable if there exist $d_l > 0$, $l = 1, \ldots, n_s$, so that

$$\begin{pmatrix} d_{l-1}(P_{l-1}A_z^{l-1} + (A_z^{l-1})^T P_{l-1}) & X_{l,l-1} \\ X_{l,l-1}^T & d_l(P_l A_z^l + (A_z^l)^T P_l) \end{pmatrix} < 0$$

for $l = 2, \ldots, n_s - 1$, where $X_{l,l-1} = 2d_l(A_z^{l,l-1})^T P_l + 2d_{l-1}P_{l-1}A_z^{l-1,l}$.

Proof. From (2) we have that $V_l = \boldsymbol{x}_l^T P_l \boldsymbol{x}_l$ is a Lyapunov function for the l-th individual subsystem, $\dot{\boldsymbol{x}}_l = \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l \boldsymbol{x}_l)$. Consider now the composite Lyapunov function $V = \sum_{l=1}^{n_s} 2d_l \boldsymbol{x}_l^T P_l \boldsymbol{x}_l$ for the distributed system (1). The derivative $\dot{V}$ can be written as

$$\begin{aligned}\dot{V} &= \sum_{l=2}^{n_s-1} 2d_l \mathcal{H}(\boldsymbol{x}_l^T P_l (A_z^l \boldsymbol{x}_l + A_z^{l,l-1}\boldsymbol{x}_{l-1} + A_z^{l,l+1}\boldsymbol{x}_{l+1}))\\ &\quad + 2d_1 \mathcal{H}(\boldsymbol{x}_1^T P_1 (A_z^1 \boldsymbol{x}_1 + A_z^{1,2}\boldsymbol{x}_2))\\ &\quad + 2d_{n_s}\mathcal{H}(\boldsymbol{x}_{n_s}^T P_{n_s}(A_z^{n_s}\boldsymbol{x}_{n_s} + A_z^{n_s,n_s-1}\boldsymbol{x}_{n_s-1}))\\ &= \sum_{l=2}^{n_s-1} 2d_l \begin{pmatrix}\boldsymbol{x}_{l-1}\\ \boldsymbol{x}_l \\ \boldsymbol{x}_{l+1}\end{pmatrix}^T\\ &\quad \cdot \begin{pmatrix} 0 & (A_z^{l,l-1})^T P_l & 0 \\ P_l A_z^{l,l-1} & \mathcal{H}(P_l A_z^l) & P_l A_z^{l,l+1} \\ 0 & (A_z^{l,l+1})^T P_l \end{pmatrix} \begin{pmatrix}\boldsymbol{x}_{l-1}\\ \boldsymbol{x}_l \\ \boldsymbol{x}_{l+1}\end{pmatrix}\\ &\quad + 2d_1 \begin{pmatrix}\boldsymbol{x}_1 \\ \boldsymbol{x}_1\end{pmatrix}^T \begin{pmatrix}\mathcal{H}(P_1 A_z^1) & (A_z^{1,2})^T P_1 \\ P_1 A_z^{1,2} & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_1 \\ \boldsymbol{x}_2\end{pmatrix}\\ &\quad + 2d_{n_s}\begin{pmatrix}\boldsymbol{x}_{n_s-1}\\ \boldsymbol{x}_{n_s}\end{pmatrix}^T\\ &\quad \cdot \begin{pmatrix} 0 & (A_z^{n_s,n_s-1})^T P_{n_s} \\ P_{n_s} A_z^{n_s,n_s-1} & \mathcal{H}(P_{n_s} A_z^{n_s})\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_{n_s-1}\\ \boldsymbol{x}_{n_s}\end{pmatrix}\end{aligned}$$

[2] We use the notation (2) as a shorthand notation for the condition $\mathcal{H}(P_l \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l) A_i^l) < 0$, which is not an LMI.

$$
\begin{aligned}
&= \sum_{l=2}^{n_s} \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \end{pmatrix}^T \\
&\quad \cdot \begin{pmatrix} d_{l-1}\mathcal{H}(P_{l-1}A_z^{l-1}) & X_{l,l-1} \\ X_{l,l-1}^T & d_l\mathcal{H}(P_lA_z^l) \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_{l-1} \\ \boldsymbol{x}_l \end{pmatrix} \\
&\quad + \boldsymbol{x}_1^T d_1 \mathcal{H}(P_1A_z^1)\boldsymbol{x}_1 \\
&\quad + \boldsymbol{x}_{n_s}^T d_{n_s} \mathcal{H}(P_{n_s}A_z^{n_s})\boldsymbol{x}_{n_s}
\end{aligned}
$$

with $X_{l,l-1} = 2d_l(A_z^{l,l-1})^TP_l + 2d_{l-1}P_{l-1}A_z^{l-1,l}$. Denote

$$
\Gamma_l = \begin{pmatrix} d_{l-1}\mathcal{H}(P_{l-1}A_z^{l-1}) & X_{l,l-1} \\ X_{l,l-1}^T & d_l\mathcal{H}(P_lA_z^l) \end{pmatrix} \tag{3}
$$

Note that for $\Gamma_2 < 0$ it is necessary that $\mathcal{H}(P_1A_z^1) < 0$, and for $\Gamma_{n_s} < 0$ it is needed that $\mathcal{H}(P_{n_s}A_z^{n_s}) < 0$. Consequently, $\dot{V} < 0$ if $\Gamma_l < 0$, for $l = 2, \ldots, n_s$. □

As already stated, it is not necessary to explicitly include the conditions

$$
\begin{aligned}
\mathcal{H}(P_1A_z^1) &< 0 \\
\mathcal{H}(P_{n_s}A_z^{n_s}) &< 0
\end{aligned}
$$

to ensure stability of the interconnected system. However, they can be included into the conditions for the second and last subsystem. Consequently, the following corollary can be formulated.

Corollary 2. The distributed TS fuzzy system with the subsystems described by (1) is globally asymptotically stable if there exist $d_l > 0$, $l = 1, \ldots, n_s$, so that

$$
\begin{pmatrix} 2d_1\mathcal{H}(P_1A_z^1) & X_{2,1} \\ X_{2,1}^T & d_2\mathcal{H}(P_2A_z^2) \end{pmatrix} < 0
$$

$$
\begin{pmatrix} d_{l-1}\mathcal{H}(P_{l-1}A_z^{l-1}) & X_{l,l-1} \\ X_{l,l-1}^T & d_l\mathcal{H}(P_lA_z^l) \end{pmatrix} < 0
$$

$$
l = 3, \ldots, n_s - 1
$$

$$
\begin{pmatrix} d_{n_s-1}\mathcal{H}(P_{n_s-1}A_z^{n_s-1}) & X_{n_s,n_s-1} \\ X_{n_s,n_s-1}^T & 2d_{n_s}\mathcal{H}(P_{n_s}A_z^{n_s}) \end{pmatrix} < 0
$$

where $X_{l,l-1} = 2d_l(A_z^{l,l-1})^TP_l + 2d_{l-1}P_{l-1}A_z^{l-1,l}$, $l = 2, \ldots, n_s$.

Note that with these modifications, the first and last conditions of Theorem 1 become less conservative.

Theorem 1 and Corollary 1 above explicitly state that the string-connected distributed TS fuzzy system is globally asymptotically stable, if each subsystem is stable and the same Lyapunov matrices can be used to ensure the stability of each pair of adjacent subsystems. However, this means that in order to verify the stability of the distributed system, one has to verify in parallel the conditions for each pair of neighboring subsystems. To develop conditions for each subsystem, i.e., circumvent the coupling, consider the condition $\Gamma_l < 0$, $l = 2, \ldots, n_s$, with Γ_l defined as in (3). This condition can be written as

$$
\begin{aligned}
&\Gamma_l = \\
&\begin{pmatrix} d_{l-1}\mathcal{H}(P_{l-1}A_z^{l-1}) + \delta I & 2d_{l-1}P_{l-1}A_z^{l-1,l} \\ 2d_{l-1}(A_z^{l-1,l})^TP_{l-1} & -\delta I \end{pmatrix} \\
&+ \begin{pmatrix} -\delta I & 2d_l(A_z^{l,l-1})^TP_l \\ 2d_lP_lA_z^{l,l-1} & d_l\mathcal{H}(P_lA_z^l) + \delta I \end{pmatrix} < 0
\end{aligned}
$$

for some $\delta > 0$. Moreover, we have

$$
\begin{aligned}
&\Gamma_{l+1} = \\
&\begin{pmatrix} d_l\mathcal{H}(P_lA_z^l) + \delta I & 2d_lP_lA_z^{l,l+1} \\ 2d_l(A_z^{l,l+1})^TP_l & -\delta I \end{pmatrix} \\
&+ \begin{pmatrix} -\delta I & 2d_{l+1}(A_z^{l+1,l})^TP_{l+1} \\ 2d_{l+1}P_{l+1}A_z^{l+1,l} & d_{l+1}\mathcal{H}(P_{l+1}A_z^{l+1}) + \delta I \end{pmatrix} < 0
\end{aligned}
$$

By imposing that both terms concerning P_l in the above expressions are negative definite, the following conditions can be formulated:

Corollary 3. The distributed TS fuzzy system with the subsystems described by (1) is globally asymptotically stable if there exist $\delta_l > 0$, $l = 1, \ldots, n_s$, so that

$$
\begin{pmatrix} \mathcal{H}(P_lA_z^l) + \delta_lI & 2P_lA_z^{l,l+1} \\ 2(A_z^{l,l+1})^TP_l & -\delta_lI \end{pmatrix} < 0
$$

$$
\begin{pmatrix} \mathcal{H}(P_lA_z^l) + \delta_lI & 2P_lA_z^{l,l-1} \\ 2(A_z^{l,l-1})^TP_l & -\delta_lI \end{pmatrix} < 0
$$

$$
l = 1, \ldots, n_s
$$

Note that in the above matrices, with a redefinition of $\delta_l = \delta/d_l$, d_l is omitted, as it is positive and it appears in all terms. Corollary 3 in fact explicitly states that each subsystem should dominate its two "incoming" interconnection terms. Moreover, at this point, one can also formulate the conditions in terms of finding P_l and δ_l, $l = 1, \ldots, n_s$.

The conditions derived so far, i.e., the conditions of Theorem 1 and Corollaries 2 and 3 are not LMIs. LMI conditions can be formulated by using e.g., the conditions of Tanaka et al. (1998), as follows.

Corollary 4. The distributed TS fuzzy system with the subsystems described by (1) is globally asymptotically stable if there exist $P_l = P_l^T > 0$ and $\delta_l > 0$, $l = 1, \ldots, n_s$, so that

$$
\begin{pmatrix} \mathcal{H}(P_lA_i^l) + \delta_lI & 2P_lA_i^{l,l+1} \\ 2(A_i^{l,l+1})^TP_l & -\delta_lI \end{pmatrix} < 0
$$

$$
\begin{pmatrix} \mathcal{H}(P_lA_i^l) + \delta_lI & 2P_lA_i^{l,l-1} \\ 2(A_i^{l,l-1})^TP_l & -\delta_lI \end{pmatrix} < 0
$$

$$
i = 1, 2, \ldots, m_l \quad l = 1, \ldots, n_s
$$

Moreover, one can also establish local stability of the interconnected system by using a fuzzy Lyapunov function and the conditions of Bernal and Guerra (2010).

4. OBSERVER DESIGN

In what follows, we extend the results developed in the previous section to observer design. For observer design, consider the distributed system consisting of n_s serially connected TS fuzzy subsystems as follows. The lth subsystem, $l = 1, 2, \ldots, n_s$ is given by:

$$
\begin{aligned}
\dot{\boldsymbol{x}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l\boldsymbol{x}_l + A_i^{l,l-1}\boldsymbol{x}_{l-1} + A_i^{l,l+1}\boldsymbol{x}_{l+1}) \\
\boldsymbol{y}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)C_i^l\boldsymbol{x}_l
\end{aligned} \tag{4}
$$

with $A_i^{1,0} = 0$, $i = 1, 2, \ldots, m_1$, and $A_i^{n_s,n_s+1} = 0$, $i = 1, 2, \ldots, m_{n_s}$.

For the simplicity of computations, in this paper we assume that the measurements do not depend on the states of the other subsystems, and that the scheduling variables $\boldsymbol{z}_l$ are measured, i.e., they do not depend on unmeasured states. Then, one can consider the observer

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l)(A_i^l \widehat{\boldsymbol{x}}_l + A_i^{l,l-1}\widehat{\boldsymbol{x}}_{l-1} \\ &\quad + A_i^{l,l+1}\widehat{\boldsymbol{x}}_{l+1} + L_i(\boldsymbol{y}_l - \widehat{\boldsymbol{y}}_l)) \\ \widehat{\boldsymbol{y}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l) C_i^l \widehat{\boldsymbol{x}}_l\end{aligned} \tag{5}$$

for the lth subsystem, $l = 1, 2, \ldots, n_\text{s}$. Under the assumption that the estimates of the neighboring subsystems are communicated, the error dynamics of the lth subsystem can be derived as

$$\begin{aligned}\dot{\boldsymbol{e}}_l &= \sum_{i=1}^{m_l} w_i^l(\boldsymbol{z}_l) \sum_{j=1}^{m_l} w_j^l(\boldsymbol{z}_l)((A_i^l - L_i^l C_j^l)\boldsymbol{e}_l \\ &\quad + A_i^{l,l-1}\boldsymbol{e}_{l-1} + A_i^{l,l+1}\boldsymbol{e}_{l+1}) \\ &= (A_z^l - L_z^l C_z^l)\boldsymbol{e}_l + A_z^{l,l-1}\boldsymbol{e}_{l-1} + A_z^{l,l+1}\boldsymbol{e}_{l+1}\end{aligned}$$

for $l = 1, 2, \ldots, n_\text{s}$.

Note that for this error dynamics, the results from Section 3 are directly applicable, and the following result can be formulated:

Corollary 5. The estimation error dynamics when the observer (5) is used for the distributed TS fuzzy system (4) are globally asymptotically stable if there exist $P_l = P_l^T > 0$, L_i^l, and $\delta_l > 0$, $i = 1, 2, \ldots, m_l$, $l = 1, 2, \ldots, n_\text{s}$, so that

$$\begin{pmatrix}\mathcal{H}(P_l A_z^l - P_l L_z^l C_z^l) + \delta_l I & 2P_l A_z^{l,l+1} \\ 2(A_z^{l,l+1})^T P_l & -\delta_l I\end{pmatrix} < 0$$
$$\begin{pmatrix}\mathcal{H}(P_l A_z^l - P_l L_z^l C_z^l) + \delta_l I & 2P_l A_z^{l,l-1} \\ 2(A_z^{l,l-1})^T P_l & -\delta_l I\end{pmatrix} < 0$$
$$l = 1, \ldots, n_\text{s}$$

To develop sufficient LMI conditions that ensure the conditions of Corollary 5, several relaxations of the double sums can be used, including the results of Wang et al. (1996); Tanaka et al. (1998); Kim and Lee (2000); Bergsten et al. (2001); Tuan et al. (2001); Guerra and Vermeiren (2004). For instance, using the results of Tuan et al. (2001), the following LMI conditions can be developed, which, when satisfied, ensure that the conditions of Corollary 5 are satisfied.

Corollary 6. The estimation error dynamics when the observer (5) is used for the distributed TS fuzzy system (4) is globally asymptotically stable if there exist $P_l = P_l^T > 0$, M_i^l, and $\delta_l > 0$, $i = 1, 2, \ldots, m_l$, $l = 1, 2, \ldots, n_\text{s}$, so that

$$\begin{aligned}&\Upsilon_{ii}^{l+} < 0 \\ &\Upsilon_{ii}^{l-} < 0 \\ &\frac{2}{m_l - 1}\Upsilon_{ii}^{l+} + \Upsilon_{ij}^{l+} + \Upsilon_{ji}^{l+} < 0 \\ &\frac{2}{m_l - 1}\Upsilon_{ii}^{l-} + \Upsilon_{ij}^{l-} + \Upsilon_{ji}^{l-} < 0\end{aligned}$$

for $i = 1, 2, \ldots, m_l$, $j = 1, 2, \ldots, m_l$, $i \neq j$, $l = 1, 2, \ldots, n_\text{s}$, with

$$\Upsilon_{ij}^{l+} = \begin{pmatrix}\mathcal{H}(P_l A_i^l - M_i^l C_j^l) + \delta_l I & 2P_l A_i^{l,l+1} \\ 2(A_i^{l,l+1})^T P_l & -\delta_l I\end{pmatrix}$$
$$\Upsilon_{ij}^{l-} = \begin{pmatrix}\mathcal{H}(P_l A_i^l - M_i^l C_j^l) + \delta_l I & 2P_l A_i^{l,l-1} \\ 2(A_i^{l,l-1})^T P_l & -\delta_l I\end{pmatrix}$$

The observer gains are recovered as $L_i^l = P_l^{-1} M_i^l$.

5. EXAMPLE

In this section we illustrate the proposed observer design method on a numerical example.

In a water recirculation system, several cascaded tanks systems are connected in a string. For instance, a system with three subsystems is shown in Figure 2.

In case of the system presented in Figure 2, water is pumped into the upper tanks 1, 3, and 5. From these tanks, the water flows to the lower tanks 2, 4, and 6. From the lower tanks, part of the water flows into a reservoir, and part is redistributed to the neighboring tanks. Each cascaded tank system has one control input u_i, which is the voltage applied to the motor of the corresponding pump, and one measured output: the water level in the lower tank. The measured outputs for the whole system are therefore h_2, h_4, and h_6. The flow rates $F_{\text{in},i}$, provided by the pumps, and the water levels in the upper tanks have to be estimated, and therefore, an observer has to be designed. The interconnection between the subsystems consists of redistributing part of the water that would flow to the reservoir to the neighboring tanks, indicated in Figure 2 by the links d_{12}, d_{21}, d_{23}, and d_{32}.

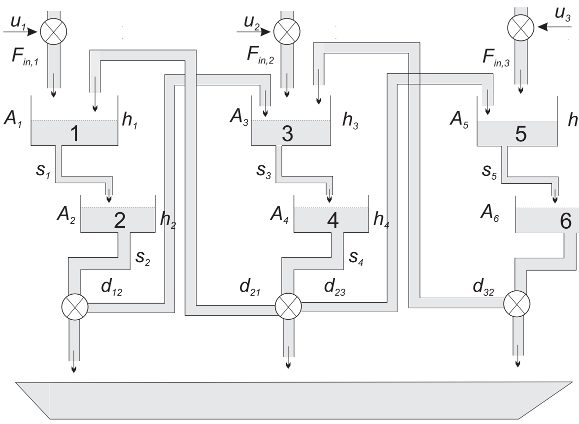

Fig. 2. Coupled cascaded tanks system.

The dynamics of the distributed system in Figure 2 are given by

$$\begin{aligned}\tau_1 \dot{F}_{\text{in},1} &= -F_{\text{in},1} + Q_{s,1} \cdot u_1 \\ \dot{h}_1 &= \frac{F_{\text{in},1}}{A_1} - \frac{s_1\sqrt{2gh_1}}{A_1} + d_{21}\frac{s_4\sqrt{2gh_4}}{A_1} \\ \dot{h}_2 &= \frac{s_1\sqrt{2gh_1}}{A_2} - \frac{s_2\sqrt{2gh_2}}{A_2}\end{aligned} \tag{6}$$

$$\tau_2\dot{F}_{\text{in},2} = -F_{\text{in},2} + Q_{s,2}\cdot u_2$$
$$\dot{h}_3 = \frac{F_{\text{in},2}}{A_3} - \frac{s_3\sqrt{2gh_3}}{A_3} + d_{12}\frac{s_2\sqrt{2gh_2}}{A_3} + d_{32}\frac{s_6\sqrt{2gh_6}}{A_3}$$
$$\dot{h}_4 = \frac{s_3\sqrt{2gh_3}}{A_4} - \frac{s_4\sqrt{2gh_4}}{A_4}$$

$$\tau_3\dot{F}_{\text{in},3} = -F_{\text{in},3} + Q_{s,3}\cdot u_3$$
$$\dot{h}_5 = \frac{F_{\text{in},3}}{A_5} - \frac{s_5\sqrt{2gh_5}}{A_5} + d_{23}\frac{s_4\sqrt{2gh_4}}{A_5}$$
$$\dot{h}_6 = \frac{s_5\sqrt{2gh_5}}{A_6} - \frac{s_6\sqrt{2gh_6}}{A_6}$$

where $d_{21}\frac{s_4\sqrt{2gh_4}}{A_1}$, $d_{12}\frac{s_2\sqrt{2gh_2}}{A_3}$, $d_{32}\frac{s_6\sqrt{2gh_6}}{A_3}$, and $d_{23}\frac{s_4\sqrt{2gh_4}}{A_5}$ represent the interconnection terms between the subsystems, i.e., the amount of water redistributed among the tanks.

The parameters are presented in Table 1, and their values are: $g = 9.81$, $A_1 = 10$, $A_2 = 11$, $A_3 = 11$, $A_4 = 9$, $A_5 = 12$, $A_6 = 10$, $s_1 = 0.2$, $s_2 = 0.1$, $s_3 = 0.2$, $s_4 = 0.125$, $s_5 = 0.25$, $s_6 = 0.135$, $Q_{s1} = Q_{s2} = Q_{s3} = 33.3$, $\tau_1 = \tau_2 = \tau_3 = 3$, $d_{12} = 0.8$, $d_{21} = 0.3$, $d_{23} = 0.6$, $d_{32} = 0.3$, $h_{\min} = 0.2$, $h_{\max} = 2$.

Table 1. Parameter values.

Parameter	Symbol	Units
Acceleration due to gravity	g	m/s^2
Cross-sectional area of tanks	A_i, $i = 1, 2, \ldots, 6$	m^2
Outlet area of tanks	s_i, $i = 1, 2, \ldots, 6$	m^2
Input to flow gains	Q_{s1}, Q_{s2}, Q_{s3}	$\text{m}^3/\text{s/V}$
Motor time constants	τ_1, τ_2, τ_3	s
Distribution ratios	d_{12}, d_{21}, d_{23}, d_{31}	–
Minimum water level	$h_{\min}$	m
Maximum water level	$h_{\max}$	m

Our goal is to design observers to estimate the flow rates $F_{\text{in},i}$, $i = 1, 2, 3$, and the water level in the upper tanks h_1, h_3, and h_5.

It is assumed that the tanks have the same height, $h_{\max} = 2\,\text{m}$, and the water level in the tanks cannot drop below $h_{\min} = 0.2\,\text{m}$. Therefore, all levels are bounded, $h_i \in [h_{\min}, h_{\max}]$, $i = 1, 2, \ldots, 6$.

In order to use the proposed design, first a TS representation of the system (6) is constructed, using the sector nonlinearity approach (Ohtake et al., 2001). We present here only the model for the first subsystem. The individual dynamics, i.e., the dynamics without the interconnection term $d_{21}\frac{s_4\sqrt{2gh_4}}{A_1}$ are represented by the convex combination of four local linear models, with the local matrices[3]

$$A_1^1 = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.19 & 0 \\ 0 & 0.18 & -0.09 \end{pmatrix} \quad A_2^1 = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.19 & 0 \\ 0 & 0.18 & -0.03 \end{pmatrix}$$
$$A_3^1 = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.06 & 0 \\ 0 & 0.05 & -0.09 \end{pmatrix} \quad A_4^1 = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.06 & 0 \\ 0 & 0.05 & -0.03 \end{pmatrix}$$

scheduling variables h_1 and h_2, weighting functions $\eta_1^0 = \frac{\sqrt{0.2}}{\sqrt{h_1}}\frac{\sqrt{2}-\sqrt{h_1}}{\sqrt{2}-\sqrt{0.2}}$, $\eta_1^1 = 1 - \eta_1^0$, $\eta_2^0 = \frac{\sqrt{0.2}}{\sqrt{h_2}}\frac{\sqrt{2}-\sqrt{h_2}}{\sqrt{2}-\sqrt{0.2}}$, $\eta_2^1 = 1 - \eta_2^0$, and membership functions $w_1 = \eta_1^0\eta_2^0$, $w_2 = \eta_1^0\eta_2^2$, $w_3 = \eta_1^2\eta_2^0$, and $w_4 = \eta_1^2\eta_2^2$. Since the measurement is linear, we have a common C matrix, $C = (0\ 0\ 1)$. The interconnection term $d_{21}\frac{s_4\sqrt{2gh_4}}{A_1}$ can be represented by the convex combination of two local models, with $A_1^{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0831 \\ 0 & 0 & 0 \end{pmatrix}$ and $A_2^{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.0083 \\ 0 & 0 & 0 \end{pmatrix}$ and weighting functions $\eta_3^0 = \frac{\sqrt{0.2}}{\sqrt{h_4}}\frac{\sqrt{2}-\sqrt{h_4}}{\sqrt{2}-\sqrt{0.2}}$, $\eta_4^1 = 1 - \eta_4^0$.

Similar representations are constructed for the other two subsystems and the remaining interconnection terms. To design the observer for the first subsystem, the following LMI problem is solved[4]: *find* $P_1 = P_1^T > 0$, M_i^1, $i = 1, 2, 3, 4$ *so that*

$$\begin{pmatrix} \mathcal{H}(P_1A_i^1 - M_i^1C) + \delta_1 I & 2P_1A_1^{1,2} \\ 2(A_1^{1,2})^TP_1 & -\delta_1 I \end{pmatrix} < 0$$
$$\begin{pmatrix} \mathcal{H}(P_1A_i^1 - M_i^1C) + \delta_1 I & 2P_1A_1^{1,2} \\ 2(A_2^{1,2})^TP_1 & -\delta_1 I \end{pmatrix} < 0$$
$$i = 1, 2, 3, 4$$

Note that due to the common measurement matrix, the double sum relaxations cannot be used. We obtain $P_1 = \begin{pmatrix} 79.55 & 1.39 & -9.76 \\ 1.39 & 12.73 & -65.94 \\ -9.76 & -65.94 & 499.59 \end{pmatrix}$, $\delta_1 = 4$, and the observer gains

$$L_1^1 = \begin{pmatrix} -0.07 \\ 24.88 \\ 3.27 \end{pmatrix} \quad L_2^1 = \begin{pmatrix} -0.07 \\ 24.88 \\ 3.33 \end{pmatrix}$$
$$L_3^1 = \begin{pmatrix} -0.04 \\ 7.17 \\ 0.94 \end{pmatrix} \quad L_4^1 = \begin{pmatrix} -0.04 \\ 7.17 \\ 1.00 \end{pmatrix}$$

The observers for the other two subsystems are designed in a similar manner. A trajectory of the estimation error for the second subsystem is presented in Figure 3. As can be seen, the observer correctly estimates the true states. This trajectory has been obtained for randomly generated inputs, drawn from the uniform distribution $\mathcal{U}[0, 0.1]$, with the true initial states being $\boldsymbol{x}_0^1 = (0.1\ 0.5\ 0.3)^T$ (first subsystem), $\boldsymbol{x}_0^2 = (2\ 0.3\ 1.4)^T$ (second subsystem), $\boldsymbol{x}_0^3 = (1\ 0.25\ 0.6)^T$ (third subsystem), and the estimated initial states being $\hat{\boldsymbol{x}}_0 = (0\ 0.2\ 0.2)^T$ for all three subsystems. For numerical integration, the *ode23* Matlab function was used.

Note that in the observer design we did not take into account that one of the scheduling variables for each subsystem, h_1, h_3, and h_5, respectively, is a state that has to be estimated. However, in the simulation, in the observers' membership functions, the estimated states were used. As can be seen, the observer still correctly estimate the states. This is due to the fact that the initial estimated states are close enough to the true initial states.

6. DISCUSSION

In this paper, we have considered stability analysis and observer design for TS fuzzy systems connected in a string. Sufficient stability and observer design conditions have

[3] All values are rounded to two decimal places.

[4] For solving LMI problems, the SeDuMi solver within the Yalmip toolbox for Matlab was used.

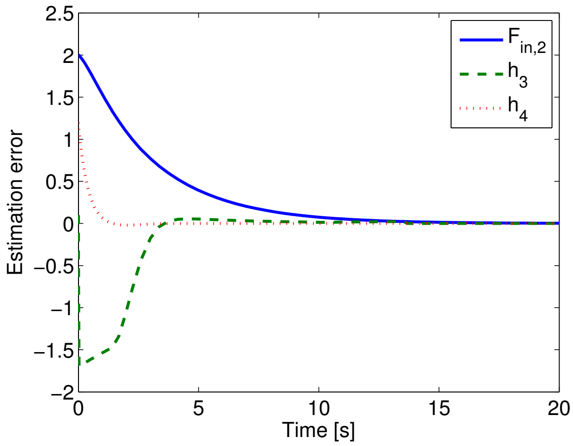

Fig. 3. Estimation error for the second subsystem.

been derived and LMI conditions have also been formulated. The application of the proposed design method has been illustrated on a numerical example.

The proposed stability analysis and observer design method can be applied either sequentially or in parallel to all the subsystems. Both types of approaches have their advantages and shortcomings. Most notably, a sequential approach (in fact the direct application of Theorem 1 or Corollary 2) has the advantage that it allows the addition of new subsystems. However, once the Lyapunov matrix P_l for the l-th subsystem has been decided upon, it is not guaranteed that it is suitable to be used for establishing the stability or to design observers for the $l + 1$st subsystem (see the condition of Theorem 1). This can be circumvented by using the analysis or design methods in parallel, in fact applying Corollaries 3 and 5.

In this paper, we considered a special class of distributed systems. In our future research, we will extend the results presented in this paper to more general, sparsely interconnected systems. Moreover, the results have been developed under two major assumptions: that each subsystem is stable, and that a composite Lyapunov function can be used for the interconnected system. While these assumptions are commonly used in the literature, they necessarily introduces conservativeness. These issues will be addressed in our future research.

REFERENCES

Akar, M. and Özgüner, U. (2000). Decentralized techniques for the analysis and control of Takagi-Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 8(6), 691–704.

Bavafa-Toosi, Y., Ohmori, H., and Labibi, B. (2006). A generic approach to the design of decentralized linear output-feedback controllers. *Systems & Control Letters*, 55(4), 282–292.

Bergsten, P., Palm, R., and Driankov, D. (2001). Fuzzy observers. In *Proceedings of the 10th IEEE International Conference on Fuzzy Systems*, volume 2, 700–703. Melbourne, Australia.

Bergsten, P., Palm, R., and Driankov, D. (2002). Observers for Takagi-Sugeno fuzzy systems. *IEEE Transactions on Systems, Man and Cybernetics, Part B*, 32(1), 114–121.

Bernal, M. and Guerra, T.M. (2010). Generalized non-quadratic stability of continuous-time Takagi-Sugeno models. *IEEE Transactions on Fuzzy Systems*, 18(4), 815–822.

Fantuzzi, C. and Rovatti, R. (1996). On the approximation capabilities of the homogeneous Takagi-Sugeno model. In *Proceedings of the Fifth IEEE International Conference on Fuzzy Systems*, 1067–1072. New Orleans, LA, USA.

Guerra, T.M. and Vermeiren, L. (2004). LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form. *Automatica*, 40(5), 823–829.

Haijun, G., Tianping, Z., and Qikun, S. (2006). Decentralized model reference adaptive sliding mode control based on fuzzy model. *Journal of Systems Engineering and Electronics*, 17(1), 182–186.

Kim, E. and Lee, H. (2000). New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 8(5), 523–534.

Krishnamurthy, P. and Khorrami, F. (2003). Decentralized control and disturbance attenuation for large-scale nonlinear systems in generalized output-feedback canonical form. *Automatica*, 39(11), 1923–1933.

Liu, S.J., Zhang, J.F., and Jiang, Z.P. (2007). Decentralized adaptive output-feedback stabilization for large-scale stochastic nonlinear systems. *Automatica*, 43(2), 238–251.

Liu, X. and Zhang, H. (2005). Stability analysis of uncertain fuzzy large-scale system. *Chaos, Solitons & Fractals*, 25(5), 1107–1122.

Ohtake, H., Tanaka, K., and Wang, H. (2001). Fuzzy modeling via sector nonlinearity concept. In *Proceedings of the Joint 9th IFSA World Congress and 20th NAFIPS International Conference*, volume 1, 127–132. Vancouver, Canada.

Palm, R. and Bergsten, P. (2000). Sliding mode observer for a Takagi-Sugeno fuzzy system. In *Proceedings of the 9th IEEE International Conference on Fuzzy Systems*, volume 2, 665–670. San Antonio, TX, USA.

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its applications to modeling and control. *IEEE Transactions on Systems, Man, and Cybernetics*, 15(1), 116–132.

Tanaka, K., Ikeda, T., and Wang, H. (1998). Fuzzy regulators and fuzzy observers: relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, 6(2), 250–265.

Tanaka, K. and Wang, H. (1997). Fuzzy regulators and fuzzy observers: a linear matrix inequality approach. In *Proceedings of the 36th IEEE Conference on Decision and Control*, volume 2, 1315–1320. San Diego, CA, USA.

Tuan, H., Apkarian, P., Narikiyo, T., and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on Fuzzy Systems*, 9(2), 324–332.

Wang, H., Tanaka, K., and Griffin, M. (1996). An approach to fuzzy control of nonlinear systems: stability and design issues. *IEEE Transactions on Fuzzy Systems*, 4(1), 14–23.

Wang, W.J. and Lin, W.W. (2005). Decentralized PDC for large-scale T-S fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 13(6), 779–786.

Xu, D., Li, Y., and Wu, T. (2006). Stability analysis of large-scale nonlinear systems with hybrid models. In *Proceedings of the Sixth World Congress on Intelligent Control and Automation*, volume 1, 1196–1200. Dalian, China.

Zhang, H., Li, C., and Liao, X. (2006). Stability analysis and H_∞ controller design of fuzzy large-scale systems based on piecewise Lyapunov functions. *IEEE Transactions on Systems, Man and Cybernetics, Part B*, 36(3), 685–698.