Local stability of discrete-time TS fuzzy systems ⋆

Zsófia Lendek * **Jimmy Lauber** **

* *Department of Automation, Technical University of Cluj-Napoca, Memorandumului 28, 400114 Cluj-Napoca, Romania, e-mail: zsofia.lendek@aut.utcluj.ro.*
** *University of Valenciennes and Hainaut-Cambresis, LAMIH, Le Mont Houy, 59313 Valenciennes Cedex 9, France, e-mail:jlauber@univ-valenciennes.fr.*

Abstract: This paper considers local stability analysis of discrete-time Takagi-Sugeno fuzzy systems, for which classically in the TS literature only global stability is considered. Using a common quadratic and nonquadratic Lyapunov function, respectively, LMI conditions are developed to establish local stability of an equilibrium point. An estimate of the region of attraction of this point is also determined. The developed conditions are illustrated on a numerical example.

Keywords: discrete-time fuzzy system; local stability; Lyapunov stability

1. INTRODUCTION

Takagi-Sugeno (TS) fuzzy models (Takagi and Sugeno, 1985) are nonlinear systems represented as convex combinations of local linear models on a compact set of the state-space. Such a representation usually facilitates the automatic stability analysis or controller design of the nonlinear system.

For the analysis of TS models the direct Lyapunov approach has been used. Using initially quadratic Lyapunov functions (Tanaka et al., 1998; Tanaka and Wang, 2001; Sala et al., 2005), later on piecewise continuous Lyapunov functions (Johansson et al., 1999; Feng, 2004), and more recently, nonquadratic Lyapunov functions (Guerra and Vermeiren, 2004; Kruszewski et al., 2008; Mozelli et al., 2009), in general linear matrix inequality (LMI) conditions are developed, which can be solved using available convex optimization methods.

The initial results, in particular those involving common quadratic Lyapunov functions, develop conditions that, when satisfied, imply the global stability of the TS model. This in fact means that any trajectory starting in the largest Lyapunov level set included in the considered compact set of the state-space will converge. In the case of the continuous-time TS models, with the introduction of nonquadratic Lyapunov functions, the developments involve the derivatives of the membership functions. Due to this, local stability results have been obtained, with the domain given by the bounds on the derivatives (Tanaka et al., 2003; Guerra and Bernal, 2009; Mozelli et al., 2009), usually being translated into bounds on the states.

In the discrete-time case, since the variation of the Lyapunov function does not involve any derivatives and thus further conditions, non-quadratic Lyapunov functions have shown a real improvement (Guerra and Vermeiren, 2004; Ding et al., 2006; Dong and Yang, 2009; Lee et al., 2011; Lendek et al., 2013, 2015) for developing global stability and design conditions. It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework. More recently, by using Polya's theorem (Montagner et al., 2007; Sala and Ariño, 2007) asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability in the sense of a chosen quadratic or nonquadratic Lyapunov function. Ding (2010) gave ANS stability conditions for both membership function-dependent model and membership function-dependent Lyapunov matrix. By increasing the complexity of the homogeneously polynomially parameter-dependent Lyapunov functions, in theory any sufficiently smooth Lyapunov function can be approximated. Unfortunately, the number of LMIs that have to be solved increase quickly, leading to numerical intractability (Zou and Yu, 2014). However, all these results involve global stability, i.e., if an equilibrium point is not globally stable, no conclusion can be drawn.

With the considerations above, in this paper, we consider the problem of establishing local stability of discrete-time TS models and estimating a domain of attraction of the equilibrium point. The structure of the paper is as follows. Section 2 presents the notations used in this paper and motivates our work through a simple example. Section 3 develops the proposed conditions for stability

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analysis, using a common quadratic and a nonquadratic Lyapunov function, respectively. The developed conditions are discussed and illustrated on a numerical example. Section 4 concludes the paper.

2. NOTATION AND PRELIMINARIES

In this paper we develop sufficient conditions for the local stability of nonlinear discrete-time systems represented by Takagi-Sugeno (TS) fuzzy models. Thus, we consider systems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r} h_i(\boldsymbol{z}(k)) A_i \boldsymbol{x}(k) \\ &= A_z \boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $\boldsymbol{x}$ denotes the state vector, r is the number of rules, $\boldsymbol{z}$ is the scheduling vector, h_i, $i = 1, 2, \ldots, r$ are normalized membership functions, and A_i, $i = 1, 2, \ldots, r$ are the local models. To motivate the research presented hereafter, consider the following example.

Example 1. Consider the nonlinear system:

$$\begin{aligned} x_1(k+1) &= x_1^2(k) \\ x_2(k+1) &= x_1(k) + 0.5 x_2(k) \end{aligned} \tag{2}$$

with $x_1(k) \in [-a, a]$, $a > 0$ being a parameter. It can be easily seen that (2) is locally stable for $x_1 \in (-1, 1)$.

The nonlinearity is x_1^2 and using the sector nonlinearity approach (Ohtake et al., 2001) on the domain $x_1(k) \in [-a, a]$, the resulting TS model is

$$\boldsymbol{x}(k+1) = h_1(x_1(k)) A_1 \boldsymbol{x} + h_2(x_1(k)) A_2 \boldsymbol{x}$$

with $h_1(x_1) = \frac{a - x_1(k)}{2a}$, $h_2(x_1(k)) = 1 - h_1(x_1(k))$, $A_1 = \begin{pmatrix} -a & 0 \\ 1 & 0.5 \end{pmatrix}$, $A_2 = \begin{pmatrix} a & 0 \\ 1 & 0.5 \end{pmatrix}$.

If $a < 1$, e.g., $a = 0.9$, the stability of the TS model can be easily proven e.g., using a common quadratic Lyapunov function.

If the sector nonlinearity approach is applied for $a > 1$, without including further conditions, no conclusion can be drawn regarding the stability of the TS model. A condition that leads to the feasibility of the associated LMI problem and thus makes it possible to draw some conclusion of local stability is e.g., $x_1^2(k) \geq 0.9 x_1^2(k+1)$. However, the question on how to obtain such a condition and its exact interpretation remains open. □

In what follows, 0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry. The subscript $z + m$ (as in A_{z+m}) stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., at $\boldsymbol{z}(k+m)$. We will also make use of the following results:

Lemma 2. (Skelton et al., 1998) Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $\mathcal{R} \in \mathbb{R}^{m \times n_x}$ such that $\text{rank}(\mathcal{R}) < n_x$. The two following expressions are equivalent:

(1) $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, \mathcal{R}\boldsymbol{x} = 0\}$
(2) $\exists \mathcal{M} \in \mathbb{R}^{m \times n_x}$ such that $Q + \mathcal{M}\mathcal{R} + \mathcal{R}^T \mathcal{M}^T < 0$ □

Lemma 3. (S-procedure) Consider matrices $F_i = F_i^T \in \mathbb{R}^{n \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, such that $\boldsymbol{x}^T F_i \boldsymbol{x} \geq 0$, $i = 1, \ldots, p$, and the quadratic inequality condition

$$\boldsymbol{x}^T F_0 \boldsymbol{x} > 0 \tag{3}$$

$\boldsymbol{x} \neq 0$. A sufficient condition for (3) to hold is: there exist $\tau_i \geq 0$, $i = 1, \ldots, p$, such that

$$F_0 - \sum_{i=1}^{p} \tau_i F_i > 0$$

. □

Analysis and design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{j=1}^{r} \sum_{k=1}^{r} h_j(\boldsymbol{z}(k)) h_k(\boldsymbol{z}(k)) \Gamma_{j,k} \boldsymbol{x} < 0 \tag{4}$$

where $\Gamma_{j,k}$, $j, k = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 4. (Wang et al., 1996) The double-sum (4) is negative, if

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i < j \end{aligned} \tag{5}$$

□

Lemma 5. (Tuan et al., 2001) The double-sum (4) is negative, if

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\frac{2}{r-1} \Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i \neq j \end{aligned} \tag{6}$$

□

3. LOCAL STABILITY ANALYSIS

Consider the TS model (1), repeated here for convenience:

$$\boldsymbol{x}(k+1) = A_z \boldsymbol{x}(k) \tag{7}$$

defined on the domain $\mathcal{D}$ including the origin.

Our goal is to develop conditions for this system to have a locally asymptotically stable equilibrium point in $\boldsymbol{x} = 0$ and determine a region of attraction. For this, let us first assume that

Assumption 6. There exists a domain $\mathcal{D}_R \subset \mathcal{D}$ and a symmetric matrix $R = R^T$ so that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathcal{D}_R$. □

Note that this assumption can always be satisfied, e.g., by reducing $\mathcal{D}_R$ to the origin.

3.1 Local quadratic stability

The following result is straightforward.

Theorem 7. The discrete-time nonlinear model (1) is locally asymptotically stable if there exist matrices $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ and scalar $\tau > 0$ so that

$$\begin{pmatrix} -P & (*) \\ M_z A_z & P - M_z - M_z^T \end{pmatrix} + \tau R < 0 \tag{8}$$

Moreover, the region of attraction , i.e., the region from which all trajectories converge to zero, includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$. □

Proof. Consider the Lyapunov function $V = \boldsymbol{x}^T(k)P\boldsymbol{x}(k)$. The difference in the Lyapunov function is

$$\Delta V = \boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}$$

In the domain $\mathcal{D}_R$ Assumption 6 holds, thus, using Proposition 3, we have $\Delta V < 0$ if there exists $\tau > 0$ so that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} + \tau \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} \leq 0$$

Furthermore, the dynamics (1) can be written as

$$(A_z \ \ -I) \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} = 0$$

Using Lemma 2, we have $\Delta V < 0$ if there exist $\mathcal{M}$ so that

$$\mathcal{M}(A_z \ \ -I) + (*) + \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} + \tau R < 0$$

Choosing $\mathcal{M} = \begin{pmatrix} 0 \\ M_z \end{pmatrix}$ leads to (8) and concludes the proof. □

Sufficient LMI conditions can easily be derived using Lemma 4 or Lemma 5, as follows.

Corollary 8. The discrete-time nonlinear model (1) is locally asymptotically stable if there exist matrices $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ and scalar $\tau > 0$ so that (5) or (6) hold, with

$$\Gamma_{i,j} = \begin{pmatrix} -P & (*) \\ M_i A_j & P - M_i - M_i^T \end{pmatrix} + \tau R < 0$$

Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$. □

Now, the question arises what happens if R is unknown and how to determine R in this case. Note that – considering (8) – if R is also a decision variable, the parameter τ does not have any effect, as P, M_i, $i = 1, 2, \ldots, r$, τ and R are all decision variables. Thus, it is possible to solve (8) in the variables P, τR, and M_i $i = 1, 2, \ldots, r$. In what follows, with a slight abuse of notation, τR will be denoted simply by R. Then the following result can be formulated.

Theorem 9. The discrete-time nonlinear model (1) is locally asymptotically stable in the domain $\mathcal{D}_S$ if there exist matrices $P = P^T > 0$, M_i, $i = 1, 2, \ldots, r$ and $R = R^T$ so that

$$\begin{pmatrix} -P & (*) \\ M_z A_z & P - M_z - M_z^T \end{pmatrix} + R < 0 \tag{9}$$

where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R \bigcap \mathcal{D}$. □

Proof. Consider the Lyapunov function $V = \boldsymbol{x}^T(k)P\boldsymbol{x}(k)$. The difference is

$$\Delta V = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}$$

Let us assume that there exists a matrix $R = R^T$ and a domain $\mathcal{D}_R$ such that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds $\forall \boldsymbol{x}(k) \in \mathcal{D}_R$. Note that since $\boldsymbol{x} = 0$ is an equilibrium point of the system (1), $\mathcal{D}_R$ always exists and it includes $\boldsymbol{x} = 0$. Then, we have $\Delta V < 0$ if

$$\Delta V < -\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} + \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} < 0$$

Writing the dynamics of (1) as

$$(A_z \ \ -I) \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} = 0$$

and using Lemma 2, we have $\Delta V < 0$ if there exist $\mathcal{M}$ so that

$$\mathcal{M}(A_z \ \ -I) + (*) + \begin{pmatrix} -P & 0 \\ 0 & P \end{pmatrix} + R < 0$$

Choosing $\mathcal{M} = \begin{pmatrix} 0 \\ M_z \end{pmatrix}$ leads to (9). Wrt. the domain of attraction, recall that (1) has been defined in the domain $\mathcal{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds in the domain $\mathcal{D}_R$. Thus, convergence is established for every trajectory starting in $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set contained in $\mathcal{D}_R \bigcap \mathcal{D}$. □

Regarding the structure of R, several possibilities can be chosen, such as, $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$, which establishes a direct relation between $\boldsymbol{x}(k)$ and $\boldsymbol{x}(k+1)$; a full R, which will give a more complex relation between two consecutive samples and thus a larger region, etc. Furthermore, depending on the structure of the system, specific structures can be chosen for R, as illustrated in the following Example.

Example 10. Recall the system in Example 1, and let us assume that the TS model is defined for $a = 2$. Note that in this case, using classical conditions, it is not possible to establish (local) stability of the model. The following options are tested:

O_1: full R. The results are presented in Figure 1(a). The resulting matrix R is

$$R = \begin{pmatrix} 1.1 & 0 & 0 & -2.31 \\ 0 & 2.7 & 0 & -1.15 \\ 0 & 0 & -2.81 & 0 \\ -2.31 & -1.15 & 0 & 0 \end{pmatrix}$$

As it can be seen, the resulting condition involve both x_1 and x_2, and the resulting set is $\mathcal{D}_S = 0$.

O_2: $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$. The results are presented in Figure 1(b). The resulting matrix R is

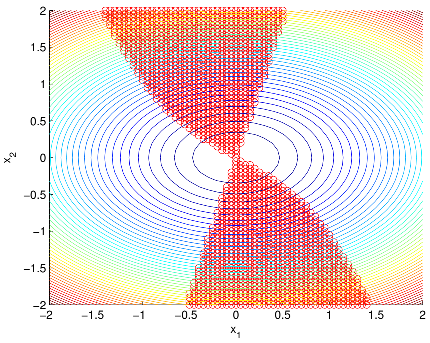
$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

i.e., it requires not only x_1 to decrease, but also x_2. This is why the resulting domain $\mathcal{D}_S = 0$.

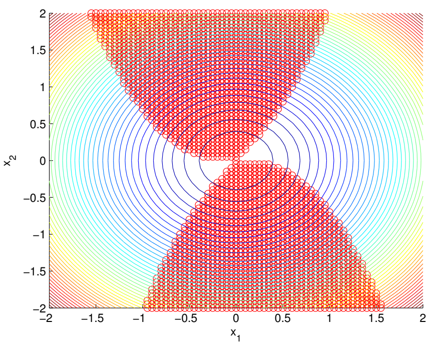
O_3: $R = \begin{pmatrix} R_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & R_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ giving the results in Figure 1(c). The resulting matrix R is

$$R = \begin{pmatrix} 0.18 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.18 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

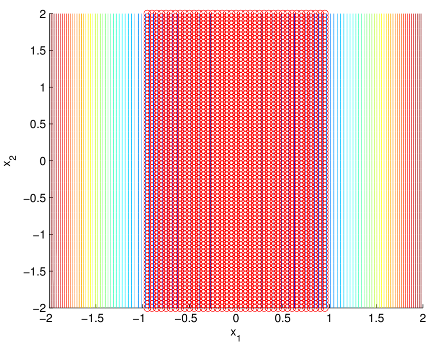
i.e., it requires only x_1 to decrease and actually gives the whole domain $\mathcal{D}_S = \{(x_1, x_2) | x_1 < 1\}$.



(a) Results using a full R as in O_1.



(b) Results using the R in O_2.



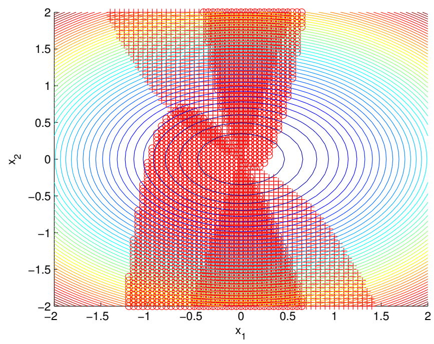
(c) Results using the specific R in O_3.

Fig. 1. Results for Example 10.

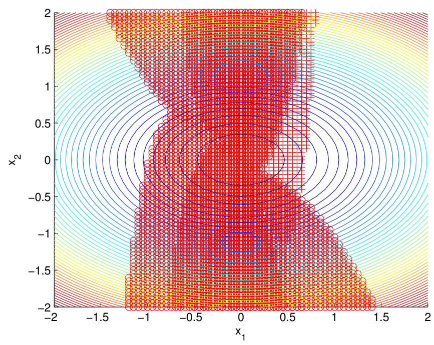
For all the cases above, in order to obtain the maximum domain $\mathcal{D}_R$, the trace of R has been maximized. In Fig. 1, the ellipses denote the Lyapunov level sets, while 'o' stands for those points that satisfy Assumption 6 with the obtained R. □

The results in Figure 1 indicate that in order to determine the maximum domain, (only) those states that contain nonlinearities should be included. On the other hand, this is a purely deterministic system, thus any trajectory that eventually gets in the domain $\mathcal{D}_S$ will converge. Therefore, let us now look at those trajectories that do not start here, but arrive in $\mathcal{D}_R$.

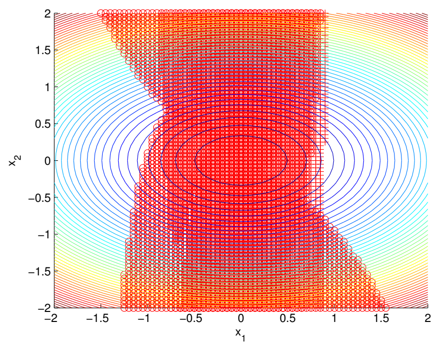
For this, consider a full R. If we check the points that are in $\mathcal{D}_P$ plus those that get to $\mathcal{D}_R$ in one step, we have for Example 10 the results in Figure 2(a), for two steps the results in Figure 2(b) and for 3 steps the results in Figure 2(c).



(a) 0 and 1 step.



(b) 0,1 and 2 steps.



(c) 0,1, 2, and 3 steps.

Fig. 2. Trajectories that converge in several steps to $\mathcal{D}_R \bigcap \mathcal{D}_P$ for Example 10.

As can be seen, in 3 steps we obtain almost the whole domain on x_1, although not on x_2. This is because the Lyapunov function is a common quadratic one and it does not include knowledge on the structure of the nonlinearities.

However, as shown above, looking to $\bigcup_{i=1}^{\beta} \mathcal{D}_{R_i}$, where $\mathcal{D}_{R_i}$ denotes the domain of those states whose trajectory will arrive in $\mathcal{D}_R$ in i steps and with β a relatively small, finite value can significantly improve the result.

Remark: Depending on the system considered, using $\mathcal{M} = [M_{1z}, M_{2z}]^T$ instead of $\mathcal{M} = [0, M_z]^T$ may improve the result. At this point, in order to reduce the number of variables involved and to be in line with results in the literature, we use $\mathcal{M} = [0, M_z]^T$.

3.2 Local nonquadratic stability

Let us now consider a nonquadratic Lyapunov function $V(\boldsymbol{x}) = \boldsymbol{x}^T(k) P_z \boldsymbol{x}(k)$. Then, similarly to the case of common quadratic Lyapunov function, the following result can be established.

Theorem 11. The equilibrium point $\boldsymbol{x} = 0$ of the discrete-time nonlinear model (1) is locally asymptotically stable if there exist matrices $P_i = P_i^T > 0$, M_i, N_i, $i = 1, 2, \ldots, r$ and scalar $\tau > 0$ so that

$$\begin{pmatrix} N_z A_z + (*) - P_z & (*) \\ M_z A_z - N_z & P_{z+1} - M_z - M_z^T \end{pmatrix} + R < 0 \tag{10}$$

where P_{z+1} denotes $\sum_{i=1}^{r} h_i(\boldsymbol{z}(k+1)) P_i$. Moreover, the region of attraction, i.e., the region from which all trajectories converge to zero, includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$. □

Proof. Consider the Lyapunov function $V = \boldsymbol{x}^T(k) P_z \boldsymbol{x}(k)$. The difference is

$$\begin{aligned} \Delta V &= \boldsymbol{x}^T(k+1) P_{z+1} \boldsymbol{x}(k+1) - \boldsymbol{x}^T(k) P_z \boldsymbol{x}(k) \\ &= \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} \end{aligned}$$

In the domain $\mathcal{D}_R$ Assumption 6 holds, thus, using Proposition 3, we have $\Delta V < 0$ if

$$\Delta V < - \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}$$

i.e.,

$$\begin{aligned} &\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} \\ &+ \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} < 0 \end{aligned}$$

Writing the dynamics of (1) as

$$\begin{pmatrix} A_z & -I \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}^T(k+1) \end{pmatrix} = 0$$

and using Lemma 2, we have $\Delta V < 0$ if there exist $\mathcal{M}$ so that

$$\mathcal{M} \begin{pmatrix} A_z & -I \end{pmatrix} + (*) + \begin{pmatrix} -P_z & 0 \\ 0 & P_{z+1} \end{pmatrix} + R < 0$$

Choosing $\mathcal{M} = \begin{pmatrix} N_z \\ M_z \end{pmatrix}$ leads to (10). Wrt. the domain of attraction, recall that (1) has been defined in the domain $\mathcal{D}$, and that

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix}^T R \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \end{pmatrix} \geq 0$$

holds in the domain $\mathcal{D}_R$. Thus, convergence is established for every trajectory starting in $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set contained in $\mathcal{D}_R \bigcap \mathcal{D}$. □

Sufficient LMI conditions can easily be derived using Lemma 4 or Lemma 5, as follows.

Corollary 12. The discrete-time nonlinear model (1) is locally asymptotically stable if there exist matrices M_i, $i = 1, 2, \ldots, r$, $P = P^T > 0$, and $R = R^T$ so that (5) or (6) hold, with

$$\Gamma_{i,j} = \begin{pmatrix} N_i A_j + (*) - P_i & (*) \\ M_i A_j - N_i & P_k - M_i - M_i^T \end{pmatrix} + R < 0$$

Moreover, the region of attraction includes $\mathcal{D}_S$, where $\mathcal{D}_S$ is the largest Lyapunov level set included in $\mathcal{D}_R$. □

Similarly to the previous case, several possibilities can be chosen for R, such as diagonal, block-diagonal, a full one or any other structure. For instance, considering Example 10 and choosing $R = \begin{pmatrix} R_1 & 0 \\ 0 & -I \end{pmatrix}$, the resulting matrix R is

$$R = \begin{pmatrix} 2 & 0.5 & 0 & 0 \\ 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This result is graphically illustrated in Figure 3. As can be seen, almost the entire domain is recovered.

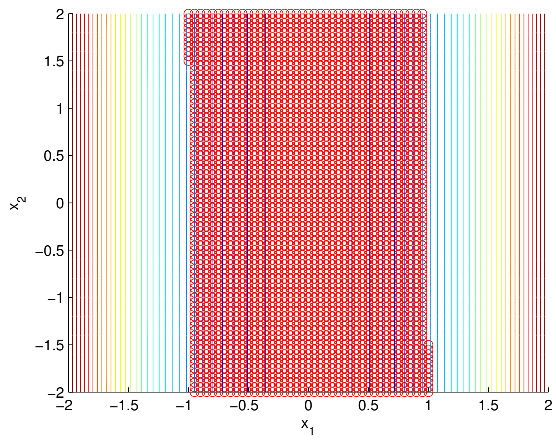


Fig. 3. Domain recovered with the conditions of Theorem 11 for Example 10.

4. CONCLUSIONS

This paper proposed conditions to establish local stability of an equilibrium point of a TS fuzzy model. To develop the conditions, first a common quadratic Lyapunov function has been considered. Results have also been given using a nonquadratic Lyapunov function and an estimate of the region of attraction has been obtained. The developed conditions were illustrated and discussed based on a numerical example. In our future work, we will investigate extensions of the proposed conditions, in particular for a nonquadratic R.

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