

# Construction of extended Lyapunov functions and control laws for discrete-time TS systems

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**Abstract**—Non-quadratic Lyapunov functions have now been more and more frequently used for the analysis and design of Takagi-Sugeno fuzzy models. In this paper, we use a delayed non-quadratic Lyapunov function to develop controller design conditions for TS systems. The conditions can easily be formulated as LMIs. We show that in certain cases the developed conditions are more relaxed than current state-of-the-art methods. In order to further reduce the conservativeness, we also extend the conditions to  $\alpha$ -sample variation.

## I. INTRODUCTION

Takagi-Sugeno (TS) fuzzy systems [1] are convex combinations of local linear models, and are able to exactly represent a large class of nonlinear systems [2].

For the analysis and design of TS models the direct Lyapunov approach has been used. Stability conditions have been derived using quadratic Lyapunov functions [3]–[5], piecewise continuous Lyapunov functions [6], [7], and nonquadratic Lyapunov functions [8]–[10] and have been in general formulated as linear matrix inequalities (LMIs). Since the development of the first stability and design conditions, the research efforts have been geared to reduce the conservativeness of the approaches.

For discrete-time TS models, non-quadratic Lyapunov functions have shown a real improvement of the design conditions [8], [11]–[13]. It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework.

Non-quadratic Lyapunov functions have been extended to double-sum Lyapunov functions in [11] and later on to polynomial Lyapunov functions in [14]–[16]. A different type of improvement in the discrete case has been developed in [9], conditions being obtained by replacing the classical one sample variation of the Lyapunov function by its variation over several samples ( $\alpha$ -sample variation).

For controller and observer design of discrete-time TS models, in [17] a non-quadratic double-sum delayed controller and observer has been used. The method of [17] has been generalized for observer design in [18], but not for controller design. In fact, the controller design method leads to an increased number of LMI problems, thereby increasing the computational costs.

In this paper, we propose a delayed nonquadratic Lyapunov function that leads to more relaxed controller design conditions, with reduced computational costs. The conditions are formulated as LMIs. We also extend the results to  $\alpha$ -sample variation [9].

The structure of the paper is as follows. Section II presents the notations used in this paper and the general form of the TS models. Section III motivates and introduces the proposed controller. Section IV develops the proposed conditions for controller design and extends them to  $\alpha$ -sample variation. Section V discusses the conditions and their applications and concludes the paper.

## II. PRELIMINARIES

The discrete-time TS model considered in this paper for controller design is

$$\mathbf{x}(k+1) = \sum_{i=1}^r h_i(\mathbf{z}(k))(A_i \mathbf{x}(k) + B_i \mathbf{u}(k)) \quad (1)$$

where  $r$  denotes the number of rules,  $A_i$  and  $B_i$ ,  $i = 1, 2, \dots, r$  are the local matrices,  $k$  is the time instant,  $\mathbf{x} \in \mathbb{R}^{n_x}$  is the state vector,  $\mathbf{u} \in \mathbb{R}^{n_u}$  is the control input,  $\mathbf{z} \in \mathbb{R}^{n_z}$  is the scheduling vector,  $h_i$ ,  $i = 1, 2, \dots, r$  denote the normalized membership functions,  $h_i \geq 0$ ,  $i = 1, 2, \dots, r$ ,  $\sum_{i=1}^r h_i(\mathbf{z}(k)) = 1$ . It is assumed that the scheduling variables  $\mathbf{z}(k)$  are available at the time instant  $k$ .

For the ease of notation, in what follows, we use

$$\mathbf{x}(k+1) = A_z \mathbf{x}(k) + B_z \mathbf{u}(k) \quad (2)$$

instead of (1), where  $A_z$  and  $B_z$  denote the convex sums  $A_z = \sum_{i=1}^r h_i(\mathbf{z}(k))A_i$ , and  $B_z = \sum_{i=1}^r h_i(\mathbf{z}(k))B_i$ , respectively. The subscript  $z$  denotes that the sum is evaluated at time instant  $k$ . The subscript  $z-$  stands for the sum being evaluated at time  $k-1$ , e.g.,  $A_{z-} = \sum_{i=1}^r h_i(\mathbf{z}(k-1))A_i$ ,  $z+$  means evaluation at time  $k+1$ , e.g.,  $A_{z+} = \sum_{i=1}^r h_i(\mathbf{z}(k+1))A_i$ ,  $z+n$  means evaluation at time  $k+n$ , e.g.,  $A_{z+n} = \sum_{i=1}^r h_i(\mathbf{z}(k+n))A_i$ , and multiple subscripts imply multiple sums, e.g.,  $A_{zz+} = \sum_{i=1}^r h_i(\mathbf{z}(k)) \sum_{j=1}^r h_j(\mathbf{z}(k+1))A_{ij}$ .

For developing the design conditions, we will make use of the following results:

**Lemma 1.** [19] Consider a vector  $\mathbf{x} \in \mathbb{R}^{n_x}$  and two matrices  $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$  and  $R \in \mathbb{R}^{m \times n_x}$  such that  $\text{rank}(R) < n_x$ . The two following expressions are equivalent:

- 1)  $\mathbf{x}^T Q \mathbf{x} < 0$ ,  $\mathbf{x} \in \{\mathbf{x} \in \mathbb{R}^{n_x}, \mathbf{x} \neq 0, R\mathbf{x} = 0\}$
- 2)  $\exists M \in \mathbb{R}^{m \times n_x}$  such that  $Q + MR + R^T M^T < 0$

Observer and controller design for TS models often lead to double-sum negativity problems of the form

$$\mathbf{x}^T \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}(k)) h_j(\mathbf{z}(k)) \Gamma_{ij} \mathbf{x} < 0 \quad (3)$$

where  $\Gamma_{ij}$ ,  $i, j = 1, 2, \dots, r$  are matrices of appropriate dimensions.

**Lemma 2.** [20] The double-sum (3) is negative, if

$$\begin{aligned} \Gamma_{ii} &< 0 \\ \Gamma_{ij} + \Gamma_{ji} &< 0, \quad i, j = 1, 2, \dots, r, i < j \end{aligned}$$

**Lemma 3.** [21] The double-sum (3) is negative, if

$$\begin{aligned} \Gamma_{ii} &< 0 \\ \frac{2}{r-1} \Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} &< 0, \quad i, j = 1, 2, \dots, r, i \neq j \end{aligned}$$

**Property 1.** (Congruence) Given a matrix  $P = P^T$  and a full column rank matrix  $Q$  it holds that

$$P > 0 \Rightarrow QPQ^T > 0$$

**Property 2.** Let  $A$  and  $B$  be matrices of appropriate dimensions and ranks, with  $B = B^T > 0$ . Then

$$(A - B)^T B^{-1} (A - B) \geq 0 \iff A^T B^{-1} A \geq A + A^T - B$$

**Property 3.** [22] (Schur complement) Consider a matrix  $M = M^T = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$ , with  $M_{11}$  and  $M_{22}$  being square matrices. Then

$$\begin{aligned} M < 0 &\iff \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \\ &\iff \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases} \end{aligned}$$

For convenience, 0 and  $I$  denote the zero and identity matrices of appropriate dimensions, and a  $(*)$  denotes the term induced by symmetry. Consequently,  $A^T P(*)$  stands for  $A^T P A$  and  $\begin{pmatrix} A & (*) \\ B & C \end{pmatrix}$  denotes  $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ .

### III. MOTIVATION

In [17], an observer design method based on the delayed Lyapunov function<sup>1</sup>  $V = e^T P_z e$  has been proposed, that improved existing conditions, without increasing the number of LMIs to be solved. In what follows, we solve the dual problem, i.e., derive controller design conditions based on a delayed Lyapunov function that are able to improve the existing conditions without increasing the number of LMIs.

<sup>1</sup> $e$  represents the estimation error in [17].

Consider the TS model (1) and the controller

$$\mathbf{u}(k) = -F_{zz-} H_{zz-}^{-1} \mathbf{x}(k) \quad (4)$$

Using the Lyapunov function  $V = \mathbf{x}^T P_z^{-1} \mathbf{x}$ , we have the difference

$$\begin{aligned} \Delta V_1 &= V(k+1) - V(k) = \\ &= \mathbf{x}(k+1)^T P_z^{-1} \mathbf{x}(k+1) - \mathbf{x}(k)^T P_z^{-1} \mathbf{x}(k) \\ &= \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \end{pmatrix}^T \begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_z^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \end{pmatrix} \end{aligned}$$

The closed-loop system dynamics are

$$(A_z - B_z F_{zz-} H_{zz-}^{-1} \quad -I) \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \end{pmatrix} = 0 \quad (5)$$

Using Lemma 1 with (5),  $\Delta V_1 < 0$ , if there exist  $M \in \mathbb{R}^{2n_x \times n_x}$  so that

$$\begin{pmatrix} -P_z^{-1} & 0 \\ 0 & P_z^{-1} \end{pmatrix} + M (A_z - B_z F_{zz-} H_{zz-}^{-1} \quad -I) + (*) < 0$$

Choosing  $M = \begin{pmatrix} 0 \\ P_z^{-1} \end{pmatrix}$  and congruence with  $\begin{pmatrix} H_{zz-} & 0 \\ 0 & P_z \end{pmatrix}$  leads to

$$\begin{pmatrix} -H_{zz-}^T P_z^{-1} H_{zz-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0$$

Using Property 2, we obtain

$$\begin{pmatrix} -H_{zz-} - H_{zz-}^T + P_z & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0 \quad (6)$$

This result can be formulated as

**Theorem 1.** The control law (4) asymptotically stabilizes the system (2) if there exist  $P_z = P_z^T$ ,  $F_{zz-}$ , and  $H_{zz-}$ , so that (6) holds.

Relaxed LMI conditions can easily be formulated using Lemmas 2 or 3, as follows.

**Corollary 1.** The closed-loop system (8) is asymptotically stable, if there exist  $P_i = P_i^T$ ,  $F_{ij}$ , and  $H_{ij}$ ,  $i, j = 1, 2, \dots, r$  so that (using Lemma 2)

$$\begin{aligned} \Gamma_{iik} &< 0 \\ \Gamma_{ijk} + \Gamma_{jik} &< 0 \end{aligned}$$

for  $i, j, k = 1, 2, \dots, r$ ,  $i > j$ , or (using Lemma 3)

$$\begin{aligned} \Gamma_{iik} &< 0 \\ \frac{2}{r-1} \Gamma_{iik} + \Gamma_{ijk} + \Gamma_{jik} &< 0 \end{aligned}$$

for  $i, j, k = 1, 2, \dots, r$ , where

$$\Gamma_{ijk} = \begin{pmatrix} -H_{jk} - H_{jk}^T + P_k & (*) \\ A_i H_{jk} - B_i F_{jk} & -P_i \end{pmatrix} < 0$$

The proof is straightforward.

Note that the conditions above are not equivalent to those in the literature, e.g., in [8], that involve the negative definiteness of sums of the form

$$\begin{pmatrix} -P_z & \\ A_z H_z - B_z F_z & -H_{z+} - H_{z+}^T + P_{z+} \end{pmatrix} (*) < 0$$

To illustrate this, consider the following example.

**Example 1.** Consider the two-rule fuzzy system

$$\mathbf{x}(k+1) = \sum_{i=1}^2 h_i(z(k))(A_i \mathbf{x}(k) + B_i \mathbf{u}(k))$$

with

$$A_1 = \begin{pmatrix} 1.5 & 2.7 \\ -1.1 & 1.8 \end{pmatrix} \quad A_2 = \begin{pmatrix} -0.4 & -0.8 \\ 0.5 & -0.8 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} -0.55 \\ 0.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this system, the conditions presented in [8], [14] (with 3 sums) are unfeasible, while using Corollary 1 we obtain:

$$P_1 = \begin{pmatrix} 1.90 & 1.26 \\ 1.26 & 1.07 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1.07 & 0.40 \\ 0.40 & 0.36 \end{pmatrix}$$

$$H_{11} = \begin{pmatrix} 1.47 & 1.00 \\ 1.01 & 0.94 \end{pmatrix} \quad H_{12} = \begin{pmatrix} 1.86 & 0.87 \\ 1.20 & 0.99 \end{pmatrix}$$

$$H_{21} = \begin{pmatrix} 0.91 & 0.31 \\ 0.34 & 0.34 \end{pmatrix} \quad H_{22} = \begin{pmatrix} 1.00 & 0.24 \\ 0.49 & 0.49 \end{pmatrix}$$

$$F_{11} = \begin{pmatrix} -0.42 & -0.28 \end{pmatrix} \quad F_{12} = \begin{pmatrix} -0.63 & -0.32 \end{pmatrix}$$

$$F_{21} = \begin{pmatrix} -0.20 & -0.13 \end{pmatrix} \quad F_{22} = \begin{pmatrix} -0.31 & -0.06 \end{pmatrix}$$

In what follows, we generalize and extend the result presented above.

#### IV. CONTROLLER DESIGN

Consider now the problem of controller design for the system (2).

In general, the controller considered is of the form

$$\mathbf{u}(k) = -F_f H_h^{-1} \mathbf{x}(k) \quad (7)$$

with  $F_f$  and  $H_h$  being (possibly) multiple sums evaluated in different time instants. At this point, let us generically denote the time at which these sums are evaluated by the subscripts  $f$  and  $h$ . The issue of how exactly these time indices should be chosen will be discussed in Section V. Note however, that these indices cannot concern future time instants, as the future scheduling variables are not available.

Using the controller (7) for the TS system (2), the closed-loop system can be expressed as

$$\mathbf{x}(k+1) = A_z \mathbf{x}(k) - B_z F_f H_h^{-1} \mathbf{x}(k) \quad (8)$$

In general, the nonquadratic Lyapunov function used for controller design is (see [8], [14], [23]) of the form  $V = \mathbf{x}(k)^T H_z^{-T} P_z H_z^{-1} \mathbf{x}(k)$ , or, for multiple sums,  $V = \mathbf{x}(k)^T H_{zz\dots z}^{-T} P_{zz\dots z} H_{zz\dots z}^{-1} \mathbf{x}(k)$ . In this paper, we use instead the Lyapunov function

$$V = \mathbf{x}(k)^T P_p^{-1} \mathbf{x}(k) \quad (9)$$

where again the subscript  $p$  stands for the time instants the (multiple) sum  $P$  will be evaluated in. Similarly to the indices in the controller gain, the problem of choosing  $p$  will be discussed in Section V. In what follows, we first derive controller design conditions and afterward consider the conditions obtained with an  $\alpha$ -sample variation [9] of the Lyapunov function.

##### A. Design conditions

Using the Lyapunov function (9) the following sufficient conditions can be formulated:

**Theorem 2.** The closed-loop system (8) is asymptotically stable, if there exist  $P_p = P_p^T$ ,  $F_f$ , and  $H_h$ , so that

$$\begin{pmatrix} -H_h - H_h^T + P_p & (*) \\ A_z H_h - B_z F_f & -P_{p+} \end{pmatrix} < 0 \quad (10)$$

The proof follows the same line as the one of Theorem 1 and is therefore not repeated here.

##### B. $\alpha$ -sample variation

Let us consider now an  $\alpha$ -sample variation [9] of the Lyapunov function. Then, the following conditions can be formulated.

**Theorem 3.** The closed-loop system (8) is asymptotically stable, if there exist  $P_p = P_p^T$ ,  $F_f$ , and  $H_h$ , so that

$$\begin{pmatrix} \Omega_{1,1} + P_p & (*) & \dots & 0 & 0 \\ \Omega_{2,1} & \Omega_{2,2} & \dots & 0 & 0 \\ 0 & \Omega_{3,2} & \dots & 0 & \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{\alpha-1,\alpha-1} & 0 \\ 0 & 0 & \dots & \Omega_{\alpha,\alpha-1} & -P_{p+\alpha} \end{pmatrix} < 0 \quad (11)$$

where  $\Omega_{i,i-1} = A_{z+i-2} H_{h+i-2} - B_{z+i-2} F_{f+i-2}$ ,  $\Omega_{i,i} = -H_{h+i-1} - H_{h+i-1}^T$ ,  $i = 1, 2, \dots, \alpha$ .

**Proof.** The  $\alpha$ -sample variation of the Lyapunov functions is

$$\begin{aligned} \Delta V_\alpha &= V(k+\alpha) - V(k) = \\ &= \mathbf{x}(k+\alpha)^T P_{p+\alpha}^{-1} \mathbf{x}(k+\alpha) - \mathbf{x}(k)^T P_p^{-1} \mathbf{x}(k) \\ &= \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \\ \vdots \\ \mathbf{x}(k+\alpha) \end{pmatrix}^T \\ &\quad \cdot \begin{pmatrix} -P_p^{-1} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & P_{p+\alpha}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \\ \vdots \\ \mathbf{x}(k+\alpha) \end{pmatrix} \end{aligned}$$

The closed-loop system dynamics for  $\alpha$  consecutive samples are

$$\begin{pmatrix} \Gamma_0 & -I & 0 & \dots & 0 & 0 \\ 0 & \Gamma_1 & -I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \Gamma_{\alpha-1} & -I \end{pmatrix} \begin{pmatrix} \mathbf{x}(k) \\ \mathbf{x}(k+1) \\ \vdots \\ \mathbf{x}(k+\alpha) \end{pmatrix} = 0 \quad (12)$$

where  $\Gamma_i = A_{z+i} - B_{z+i}F_{f+i}(H_{h+i})^{-1}$ ,  $i = 0, 1, \dots, \alpha - 1$ .

To obtain LMI conditions and to be able to recover classical conditions, one can choose for instance

$$M = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ H_{h+1}^{-T} & 0 & \dots & 0 & 0 \\ 0 & H_{h+2}^{-T} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & H_{h+\alpha-1}^{-T} & 0 \\ 0 & 0 & \dots & 0 & P_{p+\alpha}^{-1} \end{pmatrix}$$

Using Lemma 1, congruence with the matrix  $\text{diag}(H_{h+1}, H_{h+2}, \dots, H_{h+\alpha-1}, P_{p+\alpha})$ , and applying Property 2 leads directly to (11).  $\square$

## V. DISCUSSION

Note that up to this point we made no assumptions on how the sums or the number of sums used in the controller gains or in the Lyapunov function should be chosen. First, let us illustrate the developed conditions if the indices  $f$ ,  $h$ , and  $p$  are given.

Consider the fuzzy system

$$\begin{aligned} \mathbf{x}(k+1) &= \sum_{i=1}^r h_i(\mathbf{z}(k))(A_i \mathbf{x}(k) + B_i \mathbf{u}(k)) \\ &= A_z \mathbf{x}(k) + B_z \mathbf{u}(k) \end{aligned}$$

and let  $h = zz-$ ,  $f = zz-$ ,  $p = z - z-$ , i.e.,

$$\begin{aligned} P_{z-z-} &= \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}(k-1))h_j(\mathbf{z}(k-1))P_{ij} \\ H_{zz-} &= \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}(k))h_j(\mathbf{z}(k-1))H_{ij} \\ F_{zz-} &= \sum_{i=1}^r \sum_{j=1}^r h_i(\mathbf{z}(k))h_j(\mathbf{z}(k-1))F_{ij} \end{aligned}$$

which is a case similar to the one presented in Section III.

Then, the conditions (10) of Theorem 2 correspond to there exist  $P_{ij}$ ,  $F_{ij}$ ,  $H_{ij}$ ,  $i, j = 1, 2$  so that

$$\begin{pmatrix} -H_{zz-} - H_{zz-}^T + P_{z-z-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_{zz} \end{pmatrix} < 0$$

or

$$\begin{aligned} &\sum_{i_1=1}^r \sum_{i_2=1}^r \sum_{i_3=1}^r \sum_{i_4=1}^r h_{i_1}(\mathbf{z}(k))h_{i_2}(\mathbf{z}(k)) \\ &\cdot h_{i_3}(\mathbf{z}(k-1))h_{i_4}(\mathbf{z}(k-1)) \\ &\cdot \begin{pmatrix} -H_{i_2 i_3} - H_{i_2 i_3}^T + P_{i_3 i_4} & (*) \\ A_{i_1} H_{i_2 i_3} - B_{i_1} F_{i_2 i_3} & -P_{i_1 i_2} \end{pmatrix} < 0 \end{aligned}$$

In general, the number of sums in (10) is at most  $2|p| + |f| + |h| + 1$ , where  $|\cdot|$  denotes the cardinality. However, when the number of sums is  $2|p| + |f| + |h| + 1$ , the corresponding conditions are equivalent to using a common quadratic Lyapunov function and a PDC controller, as the LMIs that are obtained have to be solved for each possible index.

In order to reduce the number of LMIs and the conservativeness of the conditions, the indices in  $p$ ,  $f$ , and  $h$  should be chosen such that relaxations can be used. Due to the system dynamics (2), in order to use relaxations at all, both  $h$ , and  $f$  should contain  $z$ , the current time step. It can also be seen that due to the terms  $A_z H_h$  and  $B_z F_f$ , one can chose  $f = h$ , without increasing the computational costs.

To illustrate the choice of the delays, consider now the simplest case, when only one sum is used in  $P$ . Then, with the reasoning above that  $f$  and  $h$  should contain  $z$ , we have the inequality

$$\begin{pmatrix} -H_z - H_z^T + P_p & (*) \\ A_z H_z - B_z F_z & -P_{p+1} \end{pmatrix} < 0$$

which contains at least two sums if  $P$  is chosen constant. By choosing  $P$  a sum, such that the number of sums is minimum, we have 3 sums, as follows. If  $p = z$  then we have

$$\begin{pmatrix} -H_z - H_z^T + P_z & (*) \\ A_z H_z - B_z F_z & -P_{z+} \end{pmatrix} < 0$$

while if  $p$  is chosen  $p = z-$ , we have

$$\begin{pmatrix} -H_z - H_z^T + P_{z-} & (*) \\ A_z H_z - B_z F_z & -P_z \end{pmatrix} < 0$$

It can be easily seen that in the second case, i.e., when  $p = z-$ , we can also add<sup>2</sup> another dimension to  $H$  and  $F$ , giving more freedom, and without altering the number of sums in the condition. This choice will lead to the conditions

$$\begin{pmatrix} -H_{zz-} - H_{zz-}^T + P_{z-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0$$

which is exactly the case presented in Section III.

For an arbitrary number of indices in  $p$  and  $h$ , this choice generalizes to  $p = \{-1, -1, \dots, -1\}$  and  $f = h = \{0, 0, \dots, 0, -1, \dots, -1\}$ . Moreover, if  $|f| = |h| = 2|p|$ , this choice reduces the number of sums in (10) to  $2|p| + 1$ .

Theorem 2 above generalizes several results from the literature. For instance, the results of [8] are recovered by using  $p = h = f = z$  while Theorem 3 of [11] is obtained by choosing  $p = zz$ , and  $h = f = z$ .

In general, depending on the exact sums used,  $p$ ,  $f$ , and  $h$ , the conditions (10) can easily be transformed into LMIs and relaxations such as [3], [14], [20], [21] can be used.

Note that the conditions of Theorems 2, although they generalize several conditions from the literature, do not necessarily include those that use a different Lyapunov function, e.g., those of [14]. The choice of which Lyapunov function and consequently of which design conditions should be used is still undecided.

The next example compares the conditions proposed in this paper to that of [11] and [13]. To consider the simplest case, the relaxation of [20] is used on all the possible sums, but slack variables are not used. For solving the LMIs, the SeDuMi solver within the Yalmip [24] toolbox has been used.

<sup>2</sup>Note that  $H_{zz+}$  cannot be used, as the premise variables are not known in advance.

**Example 2.** Consider the two-rule TS model [8], [11] having the local matrices

$$A_1 = \begin{pmatrix} 1 & -b \\ -1 & -0.5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 5+b \\ 2b \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 1 & b \\ -1 & -0.5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 5-b \\ -2b \end{pmatrix}$$

The conditions of Theorem 3 in [11] are in fact a special case of Theorem 2 in Section IV-A, with  $p = zz$  and  $f = h = z$  (Method #1). The number of sums is 4, and the maximum value of  $b$  for which the LMIs are feasible is  $b = 1.547$ . For comparison purposes, the conditions presented in [8], with  $p = f = h = z$  (Method #2) are feasible up to  $b = 1.539$ , although they only involve 3 sums. Applying Theorem 2 of [13] with  $p = z - zz+$  and  $h = f = zz-$  (Method #3) using the relaxation of [20] and without slack variables,  $b = 1.565$  is obtained, but the conditions involve 5 sums. For the choice  $p = zz+$  and  $f = h = z$ , (Method #4) which involves 4 sums, the maximum  $b$  is  $b = 1.54$ .

Consider now the conditions proposed in this paper, in the light of the discussion above. For Theorem 2, according to the discussion above, we should choose  $p = z - z-$  and  $f = h = zz - z-$  (Method #5). Indeed, with this we can obtain  $b = 1.589$ . Moreover, by choosing  $p = z-$  and  $f = h = zz-$ , (Method #6) which will lead to only 3 sums, we get  $b = 1.553$ , a better result than that obtained by the conditions of [11].

A graphical comparison of the maximum feasible value of  $b$  obtained with the different methods is presented in Figure 1.

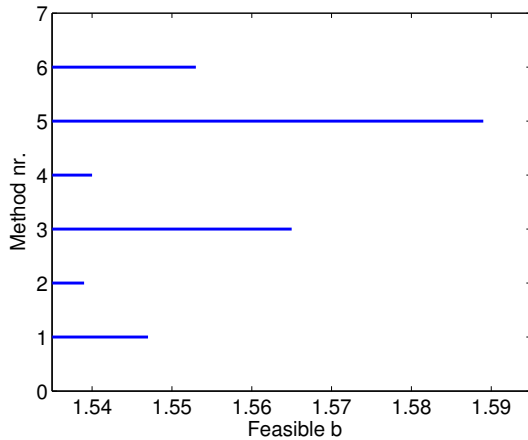


Fig. 1. Comparison of feasibility.

Let us now discuss Theorem 3. Similarly to the previous reasoning, in order to be able to use relaxations, both  $h$  and  $f$  have to contain  $z$ , i.e., the current time index. For the simplest case, when  $P$  contains only one sum, we can choose either  $p = z$  or  $p = z-$  in order to keep the number of sums in the condition (11) to a minimum. Again, it is better to use  $z-$ , and add it also to  $h$  and  $f$ , since this does not increase

the number of sums, but provides an extra degree of freedom. This means, that the condition (11) becomes

**Corollary 2.** The closed-loop system (8) is asymptotically stable, if there exist  $P_i = P_i^T$ ,  $F_{ij}$ , and  $H_{ij}$ ,  $i, j = 1, 2, \dots, r$ , so that

$$\begin{pmatrix} \Omega_{1,1} + P_{z-} & (*) & \dots & 0 & 0 \\ \Omega_{2,1} & \Omega_{2,2} & \dots & 0 & 0 \\ 0 & \Omega_{3,2} & \dots & 0 & \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{\alpha-1,\alpha-1} & 0 \\ 0 & 0 & \dots & \Omega_{\alpha,\alpha-1} & -P_{z+\alpha-1} \end{pmatrix} < 0 \quad (13)$$

where  $\Omega_{i,i-1} = A_{z+i-2}H_{z+i-2z+i-3} - B_{z+i-2}F_{z+i-2,z+i-3}$ ,  $\Omega_{i,i} = -H_{z+i-1,z+i-2} - H_{z+i-1z+i-2}^T$ ,  $i = 1, 2, \dots, \alpha$ .

Similarly to Corollary 1, LMI conditions can be stated and relaxations can be used.

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