On stabilization of discrete-time periodic TS systems

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***Abstract*—In this paper we consider controller design for periodic Takagi-Sugeno fuzzy models. For this, we use a periodic nonquadratic Lyapunov function defined at the time instants when the subsystems switch. Using the proposed conditions we are able to handle periodic Takagi-Sugeno systems where the local models or even the subsystems are unstable or cannot be stabilized. The application of the conditions is illustrated on numerical examples.**

***Index Terms*—Takagi-Sugeno fuzzy models, controller design, periodic systems, nonquadratic Lyapunov function**

I. Introduction

This paper deals with a particular class of nonlinear models with periodic parameters. Such models can be found in numerous domains such as computer control of industrial processes, or automotive, aeronautic, and aerospace industries. For instance examples related to computer control and communication systems are provided in [1], to estimate the air/fuel ratio in each cylinder on an internal combustion engine a periodic dynamic model is used in [2], [3] considers a periodic model for the rotor blades of helicopter, etc.

The stability of linear periodic systems is characterized by the monodromy transition matrix and by its eigenvalues [4]. For the stabilization of linear periodic models, results have been presented in [5]. For models including time varying delays, [6] proposed methods based on Floquet's transformation, which led to conditions for exponential stability. Results available for linear periodic systems have been extended to polytopic LPV periodic models, where the stability analysis is based on the use of quadratic [5], [7] or non-quadratic [8] Lyapunov functions.

In this paper, to represent nonlinear periodic systems, we use Takagi-Sugeno (TS) fuzzy models [9]. TS models are nonlinear, convex combinations of local linear models, and are able to exactly represent a large class of nonlinear systems [10].

For the stability analysis and controller design of TS systems the direct Lyapunov approach has been used, involving quadratic Lyapunov functions [11]–[13], piecewise continuous Lyapunov functions [14], [15], and nonquadratic Lyapunov functions [16]–[18]. In particular for discrete-time systems, non-quadratic Lyapunov functions have shown a real improvement of the design conditions [16], [19]–[21]. The stability or design conditions are generally derived in the form of linear matrix inequalities (LMIs).

Periodic TS systems [22], [23], often described by continuous dynamics and discrete dynamics as well as their interactions, have been investigated mainly in the continuous case where the stability is based on the use of a quadratic Lyapunov function [24]–[27] or a piecewise one [28], [29]. Although results are available for discrete-time linear switching systems [8], for discrete-time TS models, few results exist [30], [31].

In this paper, we consider stability analysis and controller design for discrete-time periodic TS models. To develop the conditions we use a periodic non-quadratic Lyapunov function defined in the moments when the switching takes place. The developed conditions are bilinear matrix inequalities (BMIs), which can be solved using either BMI solvers or an iterative procedure. Using the developed conditions, we are able to prove the stability of a periodic TS system having nonstable local models and stabilize a periodic system having noncontrollable local models.

The structure of the paper is as follows. Section II presents the notations and the general form of the TS models used in this paper. In Section III the proposed conditions for stability analysis are developed, discussed, and illustrated on a numerical example. Section IV extends the conditions for controller design, discusses them, and illustrates their use on a numerical example.

II. Preliminaries

In this paper we consider stability analysis and controller design of discrete-time periodic TS systems. For stability analysis, we consider subsystems of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{r_j} h_{j,i}(\boldsymbol{z}_j(k)) A_{j,i} \boldsymbol{x}(k) \\ &= A_{j,z} \boldsymbol{x}(k) \end{aligned} \tag{1}$$

and for controller design

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{r_j} h_{j,i}(\boldsymbol{z}_j(k))(A_{j,i}\boldsymbol{x}(k) + B_{j,i}\boldsymbol{u}(k)) \\ &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k)\end{aligned} \tag{2}$$

where j is the index of the subsystem, $j = 1, 2, \ldots, n_s$, n_s being the number of the subsystems, $\boldsymbol{x}$ denotes the state vector, r_j is the number of rules in the jth subsystem, $\boldsymbol{z}_j$ is the scheduling vector, $h_{j,i}$, $i = 1, 2, \ldots, r_j$ are normalized membership functions, and $A_{j,i}$ and $B_{j,i}$, $i = 1, 2, \ldots, r_j$, $j = 1, 2, \ldots, n_s$, are the local models. We assume that none of the subsystems has a finite escape time.

We consider periodic systems, i.e., the subsystems defined above are activated in a sequence $\underbrace{1, 1, \ldots, 1}_{p_1}, \underbrace{2, 2, \ldots, 2}_{p_2}, \ldots, \underbrace{n_s, n_s, \ldots, n_s}_{p_{n_s}}, \underbrace{1, 1, \ldots, 1}_{p_1}$, etc., where p_i denotes the number of samples for which the ith subsystem is active. In what follows, we will refer to p_i as the period of the ith subsystem.

0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry. The subscript $z+m$ (as in $A_{1,z+m}$) stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., $\boldsymbol{z}(k+m)$. An underlined variable $\underline{j}$ denotes the modulo of the variable, i.e., $\underline{j} = (j \mod n_s) + 1$.

In what follows, we will make use of the following results:

Lemma 1. *[32] Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{m \times n_x}$ such that rank$(R) < n_x$. The two following expressions are equivalent:*

1) $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, R\boldsymbol{x} = 0\}$
2) $\exists M \in \mathbb{R}^{n_x \times m}$ *such that* $Q + MR + R^T M^T < 0$

Controller design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k)) \Gamma_{ij} \boldsymbol{x} < 0 \tag{3}$$

where Γ_{ij}, $i, j = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 2. *[33] The double-sum* (3) *is negative, if*

$$\begin{aligned}&\Gamma_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r,\ i < j\end{aligned}$$

Lemma 3. *[34] The double-sum* (3) *is negative, if*

$$\begin{aligned}&\Gamma_{ii} < 0 \\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r,\ i \neq j\end{aligned}$$

Property 1. *(Congruence) Given a matrix $P = P^T$ and a full column rank matrix Q it holds that*

$$P > 0 \ \Rightarrow \ QPQ^T > 0$$

Property 2. *Let A and B be matrices of appropriate dimensions and ranks, with $B = B^T > 0$. Then*

$$(A - B)^T B^{-1} (A - B) \geq 0 \Longleftrightarrow A^T B^{-1} A \geq A + A^T - B$$

III. Stability analysis of periodic systems

A. Stability conditions

In this section, we consider the stability analysis of periodic TS systems of the form (1). In our previous paper [35], we have considered a switching Lyapunov function *defined only in the instants when a switching takes place* in the system, and the stability conditions have been derived based on the requirement that in each switching part, this Lyapunov function decreases. Generalizing the result in [35], this can be formulated as follows:

Theorem 1. *The periodic TS system* (1) *with periods $p_1, p_2, \ldots, p_{n_s}$ is asymptotically stable, if there exist $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$, such that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \ldots & (*) & (*) \\ MA_{1,0} & -M_{1,0} + (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & MA_{1,p_{\underline{j+1}}-1} & \Omega_{j+1,j+1} \end{pmatrix} < 0 \tag{4}$$

where[1] $MA_{1,b} = M_{\underline{j+1},z+b} A_{\underline{j+1},z}$, $M_{1,b} = M_{\underline{j+1},z+b}$, *and* $\Omega_{j+1,j+1}$ *denotes* $\Omega_{j+1,j+1} = -M_{\underline{j+1},z+p_{\underline{j+1}}-1} + (*) + P_{\underline{j+1},z+p_{\underline{j+1}}}$.

As already noted, the result above is based on the requirement that the Lyapunov function should decrease in each part. However, this is not necessary, as long as from the beginning and until the end of one complete cycle the Lyapunov function decreases, i.e., the Lyapunov function steadily decreases in each cycle. Consequently, the Lyapunov function may increase at one switch. To quantify this increase, let us impose the condition that $V(k + p_i) < \delta_i V(k)$, where $\delta_i > 0$, but not necessarily subunitary. The decrease of the Lyapunov function during the full cycle can be formulated as $V(k + \sum_{i=1}^{n_s} p_i) < \delta^\pi V(k)$, where $\delta^\pi = \prod_{i=1}^{n_s} \delta_i$, with $\delta^\pi < 1$. Then, combining it with Theorem 1, we have the following result.

Theorem 2. *The periodic TS system* (1) *with periods $p_1, p_2, \ldots, p_{n_s}$ is asymptotically stable, if there exist $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $\delta_i > 0$, $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$, such that the following conditions are satisfied:*

$$\begin{gathered}\begin{pmatrix} -\delta_j P_{j,z} & (*) & \ldots & (*) & (*) \\ MA_{1,0} & -M_{1,0} + (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & MA_{1,p_{\underline{j+1}}-1} & \Omega_{j+1,j+1} \end{pmatrix} < 0 \\ \Pi_{i=1}^{n_s} \delta_i \leq 1\end{gathered} \tag{5}$$

where $MA_{1,b} = M_{\underline{j+1},z+b} A_{\underline{j+1},z+b}$, $M_{1,b} = M_{\underline{j+1},z+b}$, *and* $\Omega_{j+1,j+1}$ *denotes* $\Omega_{j+1,j+1} = -M_{\underline{j+1},z+p_{\underline{j+1}}-1} + (*) + P_{\underline{j+1},z+p_{\underline{j+1}}}$.

Proof. *Consider the switching Lyapunov function,* defined only in the instants when a switching takes place *in the system:*

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P_{j,z} \boldsymbol{x}(k) \text{ if the active subsystem was } j$$

[1] The modulo is used for the ease of notation, as, after the last subsystem, due to the periodicity, follows the first one.

Then, the condition $V(\boldsymbol{x}(k+p_{\underline{j+1}})) < \delta_j V(\boldsymbol{x}(k))$, for some $\delta_j > 0$, can be written as

$$V(\boldsymbol{x}(k+p_{\underline{j+1}})) - \delta_j V(\boldsymbol{x}(k)) =$$

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -\delta_j P_{j,z} & 0 \\ 0 & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix}$$

The system dynamics during the $p_{\underline{j+1}}$ samples are

$$\Upsilon_{j+1} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix} = 0$$

with

$$\Upsilon_{j+1} = \begin{pmatrix} A_{\underline{j+1},z} & -I & \dots & 0 & 0 \\ 0 & A_{\underline{j+1},z+1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & A_{\underline{j+1},z+p_{\underline{j+1}}-1} & -I \end{pmatrix}$$

Using Lemma 1, $V(\boldsymbol{x}(k+p_{\underline{j+1}})) < \delta_j V(\boldsymbol{x}(k))$, if there exists M such that

$$\begin{pmatrix} -\delta_j P_{j,z} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} + M\Upsilon_{j+1} + (*) < 0$$

A particular choice of M is

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 \\ M_{\underline{j+1},z} & 0 & \dots & 0 \\ 0 & M_{\underline{j+1},z+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_{\underline{j+1},z+p_{\underline{j+1}}-1} \end{pmatrix}$$

which leads to the sufficient conditions

$$\begin{pmatrix} -\delta_j P_{j,z} & (*) & \dots & (*) & (*) \\ MA_{1,0} & -M_{1,0}+(*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & MA_{1,p_{\underline{j+1}}-1} & \Omega_{j+1,j+1} \end{pmatrix} < 0$$

$$\Pi_{i=1}^{n_s} \delta_i \leq 1$$

where $MA_{1,b} = M_{\underline{j+1},z+b} A_{\underline{j+1},z+b}$, $M_{1,b} = M_{\underline{j+1},z+b}$, and $\Omega_{j+1,j+1}$ denotes $\Omega_{j+1,j+1} = -M_{\underline{j+1},z+p_{\underline{j+1}}-1} + (*) + P_{\underline{j+1},z+p_{\underline{j+1}}}$. During one complete cycle, the above conditions amount to

$$V(\boldsymbol{x}(k+\sum_{i=1}^{n_s} p_i)) < \delta_{n_s} V(\boldsymbol{x}(k+\sum_{i=1}^{n_s-1} p_i)) <$$

$$< \delta_{n_s}\delta_{n_s-1} V(\boldsymbol{x}(k+\sum_{i=1}^{n_s-2} p_i)) < \dots$$

$$< \dots < \Pi_{i=1}^{n_s} \delta_i V(\boldsymbol{x}(k))$$

Consequently, if $\prod_{i=1}^{n_s} \delta_i \leq 1$, V is decreasing during a complete cycle, and the periodic system is asymptotically stable. □

B. Discussion and example

In this section we discuss the conditions developed above. First of all, it has to be noted that the conditions of Theorem 2 are not related to the stability of the individual subsystems. Indeed, switching between unstable subsystems can lead to a periodic system that is asymptotically stable. The introduction of the constants δ_i actually allows for the increase of the Lyapunov function while one or several subsystems are active. However, due to the condition $\prod_{i=1}^{n_s} \delta_i < 1$, during a whole cycle (be that from the first to the first or from the second to second, etc. subsystem), the Lyapunov function decreases. Even if this condition is not satisfied, $\prod_{i=1}^{n_s} \delta_i$ gives an upper bound on the *increase* of the Lyapunov function during a cycle and therefore a measure of distance from proving stability. The proposed conditions, as they are stated in Theorem 2, are unfortunately BMIs. However, the search for δ_i, $i = 1, 2, \dots, n_s$, can be done iteratively. Alternatively, one can use e.g., available BMI solvers to solve directly the BMI problem.

It is important to note that the conditions above can easily be changed to verify the *instability* of the TS model, as follows.

Theorem 3. *The equilibrium point $\boldsymbol{x} = 0$ of the periodic TS system* (1) *with periods $p_1, p_2, \dots, p_{n_s}$ is locally unstable, if there exist $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $\sigma_i > 0$, $j = 1, 2, \dots, n_s$, $i = 1, 2, \dots, r_j$, such that the following conditions are satisfied:*

$$\begin{pmatrix} \Omega & (*) & \dots & (*) & (*) \\ -M_{1,0} & MA_{1,1}+(*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -M_{1,p_{\underline{j+1}}-1} & \sigma_j P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix}$$

$$> 0$$

$$\Pi_{i=1}^{n_s} \sigma_i \leq 1 \qquad (6)$$

where $MA_{1,b} = M_{\underline{j+1},z+b} A_{\underline{j+1},z+b}$, $M_{1,b} = M_{\underline{j+1},z+b} + (*)$, and Ω denotes $\Omega = -P_{j,z} + MA_{1,0} + (*)$.

The proof follows the same lines as of Theorem 2, by imposing the condition that $\sigma_j V(\boldsymbol{x}(k+p_{\underline{j+1}})) > V(\boldsymbol{x}(k))$, and is therefore not repeated here.

Let us now illustrate the application of Theorem 2 on an example.

Example 1. *Consider a periodic TS model composed of 3 subsystems, each being active for $p_i = 2$ samples, $i = 1, 2, 3$, with the local matrices given by*

$$A_{1,1} = \begin{pmatrix} 0.23 & 0.13 \\ 0.90 & 0.75 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 0.15 & 0.30 \\ 0.40 & 0.88 \end{pmatrix}$$

$$A_{2,1} = \begin{pmatrix} -0.47 & 0.18 \\ -0.45 & 0.45 \end{pmatrix} \qquad A_{2,2} = \begin{pmatrix} 0.07 & 0.74 \\ -0.10 & -0.31 \end{pmatrix}$$

$$A_{3,1} = \begin{pmatrix} 0.40 & 0.20 \\ 0.96 & -0.43 \end{pmatrix} \qquad A_{3,2} = \begin{pmatrix} 1.21 & 0.14 \\ -0.42 & 0.41 \end{pmatrix}$$

Not all the local matrices are stable: $A_{1,2}$ and $A_{3,2}$ are unstable. However, the periodic system is stable, as indicated by the trajectories in Figure 1. For these particular trajectories, the membership functions used were $h_{1,1}(\boldsymbol{z}) = e^{-x_1^2}$,

$h_{1,2}(\boldsymbol{z}) = 1 - h_{1,1}(\boldsymbol{z})$, $h_{2,1}(\boldsymbol{z}) = \cos(x_1)^2$, $h_{2,2}(\boldsymbol{z}) = 1 - h_{2,1}(\boldsymbol{z})$, $h_{3,1}$ *has been generated from a uniform random distribution, and* $h_{3,2}(\boldsymbol{z}) = 1 - h_{3,1}(\boldsymbol{z})$. *The initial states were* $\boldsymbol{x}(0) = (-1, 1)^T$. *Solving*[2] *the conditions of Theorem 2*

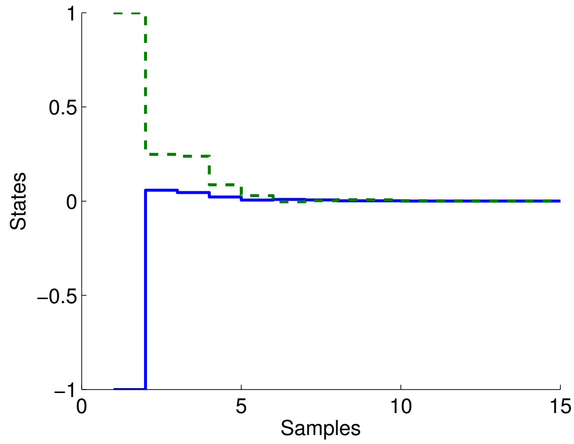

Fig. 1. Trajectories of the periodic system in Example 1.

using the relaxation 2, we obtain positive definite P*s and the following* δ*s:*

$$\delta_1 = 0.79216 \quad \delta_2 = 0.79088 \quad \delta_3 = 0.79061$$

Although in the example above we used a BMI solver to solve the conditions of Theorem 2, and therefore prove the stability of the periodic TS system, an iterative procedure can also be used as follows. It has to be noted that for large enough, fixed δs the conditions of Theorem 2 are feasible. Once such δs have been found, the conditions become LMIs and can be solved for P and M. Afterward, one can iterate by solving for δ and M and P and M, respectively.

Another possibility is to consider the α-sample variation [37] of the Lyapunov function. In this way, the necessity of solving BMIs is circumvented, as all the δs are gathered in a single one, ultimately leading to a decrease of the Lyapunov function through a cycle. However, in this way the number of the LMIs increases very much, eventually leading to unfeasibility due to the limitation of the available solvers.

IV. Controller design

A. Design conditions

In this section we extend the conditions developed in Section III-A for stability analysis for controller design. For this, we consider the periodic TS model described by (2) and a controller of the form

$$\boldsymbol{u}(k) = -F_{j,z} H_{j,z}^{-1} \boldsymbol{x}(k) \tag{7}$$

if the jth subsystem is active at sample k. The closed-loop system is then given by

$$\boldsymbol{x}(k+1) = (A_{j,z} - B_{j,z} F_{j,z} H_{j,z}^{-1}) \boldsymbol{x}(k) \tag{8}$$

[2]To solve the BMIs, the solver PenBMI [36] has been used.

if the jth subsystem is active, this again being a periodic TS system.

For (8) the following result can be formulated.

Theorem 4. *The controller* (7) *asymptotically stabilizes the periodic TS model* (7) *if there exist* $P_{j,i} = P_{j,i}^T > 0$, $H_{j,i}$, $\delta_j > 0$, $j = 1, 2, \ldots, n_s$, $i = 1, 2, \ldots, r_j$, *so that*

$$\begin{pmatrix} -\delta_j \Omega_{1,1} & (*) & \ldots & (*) & (*) \\ G_{1,0} & -H_{j,z+1} + (*) & \ldots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & G_{1,p_{\underline{j+1}}-1} & -P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0$$

$$\Pi_{i=1}^{n_s} \delta_i \leq 1$$

where $\Omega_{1,1} = H_{j,z} + H_{j,z}^T - P_{j,z}$ *and* $G_{1,b} = A_{\underline{j+1},z+b} H_{\underline{j+1},z+b} - B_{\underline{j+1},z+b} F_{\underline{j+1},z+b}$, $j = 1, 2, \ldots, n_s$.

Proof. *Similarly to stability analysis, consider a switching Lyapunov function,* defined only in the instants when a switching takes place *in the system of the form*

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P_{j,z}^{-1} \boldsymbol{x}(k) \text{ if the active subsystem was } j$$

Then, the condition $V(\boldsymbol{x}(k+p_{\underline{j+1}})) < \delta_j V(\boldsymbol{x}(k))$, *for some* $\delta_j > 0$, *can be written as*

$$V(\boldsymbol{x}(k+p_{\underline{j+1}})) - \delta_j V(\boldsymbol{x}(k)) =$$

$$\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -\delta_j P_{j,z}^{-1} & 0 \\ 0 & P_{\underline{j+1},z+p_{\underline{j+1}}}^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix}$$

The system dynamics during the $p_{\underline{j+1}}$ *samples are*

$$\Upsilon_{j+1} \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1) \\ \vdots \\ \boldsymbol{x}(k+p_{\underline{j+1}}) \end{pmatrix} = 0$$

with

$$\Upsilon_{j+1} = \begin{pmatrix} \Upsilon_{1,0} & -I & \ldots & 0 & 0 \\ 0 & \Upsilon_{1,1} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \Upsilon_{1,p_{\underline{j+1}}-1} & -I \end{pmatrix}$$

where $\Upsilon_{1,b}$ *denotes* $\Upsilon_{1,b} = A_{\underline{j+1},z+b} - B_{\underline{j+1},z+b} F_{\underline{j+1},z+b} H_{\underline{j+1},z+b}^{-1}$.

Using Lemma 1, $V(\boldsymbol{x}(k+p_{\underline{j+1}})) < \delta_j V(\boldsymbol{x}(k))$, *if there exists* M *such that*

$$\begin{pmatrix} -\delta_j P_{j,z} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} + M\Upsilon_{j+1} + (*) < 0$$

Choosing

$$M = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ H_{\underline{j+1},z+1}^{-1} & 0 & \ldots & 0 \\ 0 & H_{\underline{j+1},z+2}^{-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & P_{\underline{j+1},z+p_{\underline{j+1}}}^{-1} \end{pmatrix}$$

and congruence with

$$\begin{pmatrix} H^T_{\underline{j+1},z} & 0 & \cdots & 0 \\ 0 & H^T_{\underline{j+1},z+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix}$$

leads to

$$\begin{pmatrix} \Gamma_{1,1} & (*) & \cdots & (*) & (*) \\ G_{1,0} & -H_{j,z+1} + (*) & \cdots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & G_{1,p_{\underline{j+1}}-1} & -P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0$$
$$\Pi_{i=1}^{n_s} \delta_i \leq 1$$

where $\Gamma_{1,1} = -\delta_j H^T_{j,z} P^{-1}_{j,z} H_{j,z}$ *and* $G_{1,b} = A_{\underline{j+1},z+b} H_{\underline{j+1},z+b} - B_{\underline{j+1},z+b} F_{\underline{j+1},z+b}$, $j = 1, 2, \ldots, n_s$. *Applying Property 2 we obtain the conditions of Theorem 4. Moreover, during one complete cycle, the above conditions amount to*

$$V(\boldsymbol{x}(k + \sum_{i=1}^{n_s} p_i)) < \delta_{n_s} V(\boldsymbol{x}(k + \sum_{i=1}^{n_s-1} p_i)) <$$
$$< \delta_{n_s} \delta_{n_s-1} V(\boldsymbol{x}(k + \sum_{i=1}^{n_s-2} p_i)) < \ldots$$
$$< \ldots < \Pi_{i=1}^{n_s} \delta_i V(\boldsymbol{x}(k))$$

Consequently, if $\prod_{i=1}^{n_s} \delta_i \leq 1$, V *is decreasing during a complete cycle, and the periodic system is asymptotically stabilized.* □

B. Discussion and example

Let us now discuss the conditions developed above. Similarly to the stability conditions, it has to be noted that the conditions of Theorem 4 are not related to the controllability of the individual subsystems. Switching between subsystems that are not stabilizable on their own can lead to a periodic system that is asymptotically stabilized. Again, if the condition $\prod_{i=1}^{n_s} \delta_i < 1$ is not satisfied, $\prod_{i=1}^{n_s} \delta_i$ gives an upper bound on the *increase* of the Lyapunov function during a cycle and therefore a measure of distance from stabilization, together with a hint of which subsystem is "problematic". The proposed conditions are BMIs, therefore either a BMI solver is needed to solve them or an iterative procedure has to be employed.

Although similarly to the stability analysis, the conditions can easily be changed to prove that a periodic system cannot be stabilized, this proof would unfortunately be valid only for the control structure and Lyapunov function considered. Consequently, one would only be able to prove that there is no control law of the form (7) that stabilizes the periodic system.

To illustrate the control design using the conditions of Theorem 4 consider the following example.

Example 2. *Consider a periodic TS model composed of 2 subsystems, each being active for* $p_i = 2$ *samples,* $i = 1, 2$, *with the local matrices given by*

$$A_{1,1} = \begin{pmatrix} 0.42 & 0.73 \\ 0.89 & 0.57 \end{pmatrix} \qquad A_{1,2} = \begin{pmatrix} 0.04 & 0.56 \\ 0.67 & -0.25 \end{pmatrix}$$
$$A_{2,1} = \begin{pmatrix} -0.37 & -1.47 \\ -0.30 & -0.23 \end{pmatrix} \qquad A_{2,2} = \begin{pmatrix} 0.11 & 1.44 \\ 0.31 & -0.35 \end{pmatrix}$$
$$B_{1,1} = B_{1,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad B_{2,1} = B_{2,2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $B_{2,1} = B_{2,2} = 0$, *the local matrices of the second subsystem cannot be stabilized. However, the periodic system can be stabilized, as indicated by the trajectories in Figure 2. For these particular trajectories, the membership functions used were* $h_{1,1}(\boldsymbol{z}) = e^{-x_1^2}$, $h_{1,2}(\boldsymbol{z}) = 1 - h_{1,1}(\boldsymbol{z})$, $h_{2,1}(\boldsymbol{z}) = \cos(x_1)^2$, $h_{2,2}(\boldsymbol{z}) = 1 - h_{2,1}(\boldsymbol{z})$, *and the initial states were* $\boldsymbol{x}(0) = (5, -4)^T$. *The control input is presented in Figure 3. Since the input gains of the second subsystem are zero, this subsystem is not controlled.*

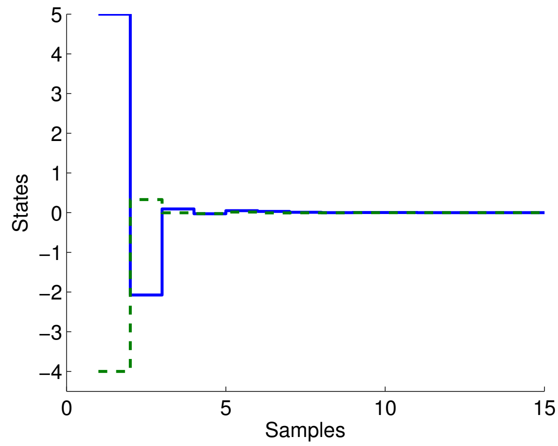

Fig. 2. Trajectories of the stabilized periodic system in Example 2.

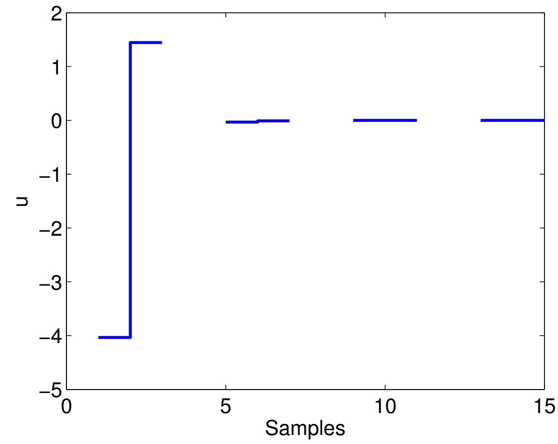

Fig. 3. Control input used to stabilize the system in Example 2.

Solving the conditions of Theorem 4 we obtain[3] *the follow-*

[3]All values are truncated to two decimal places.

ing controller gains:

$$H_{1,1} = 10^4 \begin{pmatrix} 5.68 & 0.67 \\ -0.86 & 3.51 \end{pmatrix} \qquad H_{1,2} = 10^4 \begin{pmatrix} 5.60 & 0.02 \\ -0.34 & 3.65 \end{pmatrix}$$

$$H_{2,1} = 10^4 \begin{pmatrix} 5.61 & -0.58 \\ -0.87 & 1.67 \end{pmatrix} \qquad H_{2,2} = 10^4 \begin{pmatrix} 5.54 & -0.26 \\ 0.01 & 2.06 \end{pmatrix}$$

$$F_{1,1} = 10^4 \begin{pmatrix} 4.93 \\ 3.10 \end{pmatrix} \qquad F_{1,2} = 10^4 \begin{pmatrix} 3.92 \\ -0.56 \end{pmatrix}$$

with

$$P_{1,1} = 10^4 \begin{pmatrix} 4.00 & -1.04 \\ -1.04 & 0.88 \end{pmatrix} \qquad P_{1,2} = 10^4 \begin{pmatrix} 3.90 & -0.06 \\ -0.06 & 0.93 \end{pmatrix}$$

$$P_{2,1} = 10^4 \begin{pmatrix} 6.86 & -0.22 \\ -0.22 & 2.93 \end{pmatrix} \qquad P_{2,2} = 10^4 \begin{pmatrix} 6.94 & 0.06 \\ 0.06 & 3.32 \end{pmatrix}$$

$$\delta_1 = \delta_2 = 0.5$$

Similarly to stability analysis, although we used a BMI solver to solve the conditions of Theorem 4, an iterative procedure can also be used.

While we do not present it in this paper, a similar result can be obtained for observer design, under the assumption that the scheduling variables are known. In such a case, it is possible to design an observer that is able to estimate the states of a periodic TS model of whose local models are unobservable.

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