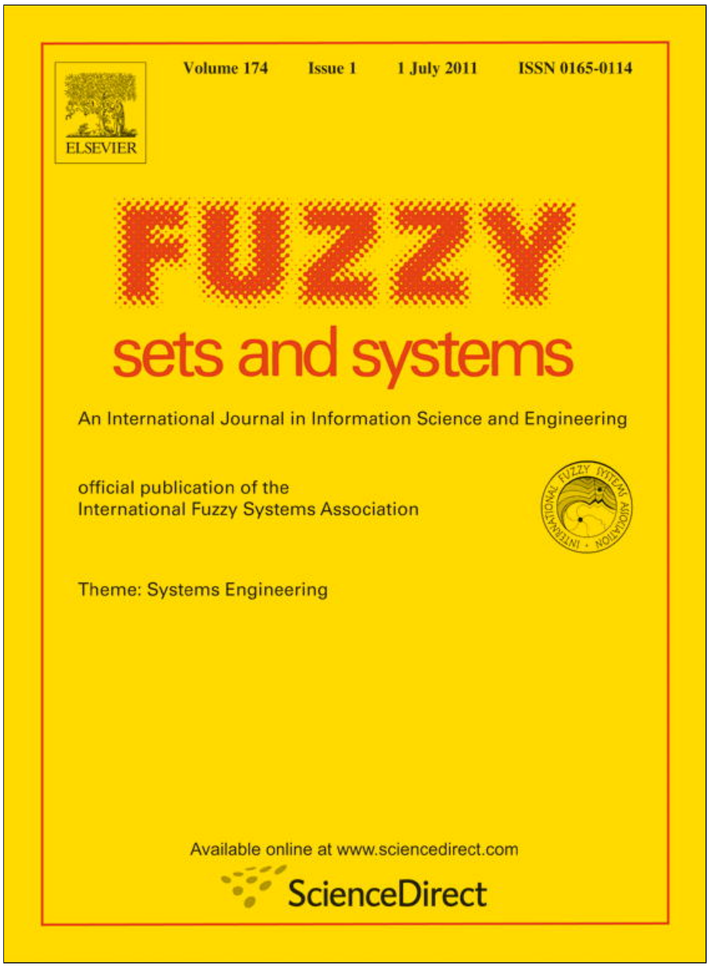
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Sequential stability analysis and observer design for distributed TS fuzzy systems

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Abstract

Many complex physical systems are the interconnection of lower-dimensional subsystems. For such systems, distributed stability analysis and observer design presents several advantages with respect to centralized approaches, such as modularity, easier analysis and design, and reduced computational complexity. Applications include distributed process control, traffic and communication networks, and economic systems. In this paper, we propose sequential stability analysis and observer design for distributed systems where the subsystems are represented by Takagi–Sugeno (TS) fuzzy models. The analysis and design are done sequentially for the subsystems, allowing for the online addition of new subsystems. The conditions are formulated as LMIs and are therefore easy to solve. The approach is illustrated on simulation examples.

Keywords: TS fuzzy systems; Distributed system; Fuzzy observers; Lyapunov stability; Distributed observers

1. Introduction

Many physical systems, such as power systems, communication networks, economic systems, traffic networks, productions systems, and water logistics are composed of interconnections of lower-dimensional subsystems. Recently, decentralized analysis and control design for such systems has received much attention [1–6]. Although in many cases the performance of the centralized design is superior [7] to that of decentralized design, there are many reasons to use a decentralized approach. For control purposes, the decentralized design presents several advantages: flexibility, fault tolerance, and simplified design and tuning. In addition, in many cases, the structure of the overall system is not fixed, i.e., subsystems may be added or removed online, and therefore a centralized analysis and/or design becomes computationally intractable.

A large class of nonlinear systems can be represented by Takagi–Sugeno (TS) fuzzy models [8], which in theory can approximate a general nonlinear system to an arbitrary degree of accuracy [9]. The TS fuzzy model consists of a fuzzy rule base. The rule antecedents partition a given subspace of the model variables into fuzzy regions, while the consequence of each rule is usually a linear or affine model, valid locally in the corresponding region.

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For a centralized fuzzy model, well-established methods and algorithms exist to analyze the stability or to design fuzzy observers. Several types of observers have been developed for continuous-time TS fuzzy systems, among which: fuzzy Thau–Luenberger observers [10,11], reduced-order observers [12,13], and sliding-mode observers [14]. Most of the stability and design conditions rely on the feasibility of an associated system of linear matrix inequalities (LMIs).

Decentralized control and estimation has received much attention [3–5,15–25] in the context of large-scale processes and distributed systems. Recently, stability analysis and decentralized control design for distributed TS systems have also been studied [5,18,26–33]. However, results for state estimation in distributed TS fuzzy systems are scarce. Although approaches for distributed stabilization and control [26,27,32] employ observers in order to estimate the states that are not directly available, these approaches [5,26,32] assume that the measurements of each subsystem refer only to the states of the considered subsystem. Moreover, if only the monitoring of a process that is not asymptotically stable is required, an observer is necessary, without a control law. The observer design in itself represents several challenges: the scheduling vector may depend on the states to be estimated; for distributed systems that are not stabilized, the interconnection terms may never converge to zero and estimated states have to be communicated; and by introducing a new subsystem into the system, the measurement matrices may change.

In this paper we consider the distributed stability analysis and observer design for a system composed of interconnected subsystems. Each subsystem is represented by a TS fuzzy model. The coupling between the subsystems is realized through their states, i.e., the states of a subsystem may influence the dynamics of another subsystem.

While in centralized stability analysis of TS fuzzy systems several types of Lyapunov functions have been employed, stability analysis of distributed TS systems mainly relies on the existence of a common quadratic Lyapunov function for each subsystem. Most results make use of the assumption that the number of subsystems and some bounds on the interconnection terms are known a priori, and the analysis of the subsystems is performed in parallel. For instance, an early result that relies on the existence of an M-matrix [1] or positive definite matrices has been formulated in [16,34]. In these approaches, LMI conditions are solved in parallel to establish the stability of the individual subsystems, and afterward the stability of the whole system is verified. For hybrid linear-fuzzy systems, a method to establish the stability of the distributed system has been proposed in [31].

An approach for distributed TS systems with affine consequents, based on piecewise Lyapunov functions has been developed in [5]. This approach is an extension of the result in [35] to distributed TS systems, but only linear interconnection terms among the subsystems are considered. Moreover, the analysis itself, although it concerns distributed systems, is not distributed, as it has to be performed at the same time in parallel for all the subsystems.

One particular type of TS systems that have been extensively investigated both in stability analysis and in (robust) control are uncertain TS fuzzy systems. For stability analysis of uncertain distributed TS systems, a result has been formulated in [2]. However, using this approach, in order to establish the stability of a distributed system, the conditions have to be verified for all the subsystems at the same time, in parallel.

In all the references above it is assumed that the structure of the system is fixed, i.e., subsystems can no longer be added to it. In these results, the stability of the whole system is established by verifying the subsystems in parallel. In this paper, we propose a method for the sequential stability analysis of distributed TS systems, that can handle the analysis of distributed TS models to which subsystems are added online.

We also consider the observer design for such TS systems. The results in the literature concerning stability analysis of distributed TS systems can be directly extended for observer design under the assumptions that (1) the scheduling vector depends only on measured variables and (2) the estimated states are communicated between the subsystems that influence each other. For observer design, the general approach is that first one constructs a set of observers for the independent subsystems. Afterward, one either incorporates an appropriate compensation to account for the influence of other subsystems or determines conditions under which the collection of the individual observers is a valid observer for the distributed system. In general, it is assumed that the measured or estimated variables are communicated between the subsystems that directly influence each other. However, the extension of the results regarding stability analysis of distributed TS systems to observer design has not been reported in the literature.

Parallel observer-based control design [33,36–40] has been considered in several settings, such as tracking control [33], adaptive control [37,39,40], robust control [36,37,41], control in the presence of time delay [38,41,42], and their

[1] A square matrix M is an M-matrix if the off-diagonal elements are all negative and all the eigenvalues of M have non-negative real part.

combinations. However, in all these results, the observer and the controller have to be designed simultaneously. Without a stabilizing state-feedback controller, the convergence to zero of the estimation error cannot be guaranteed.

The contribution of this paper is twofold: first, we consider stability analysis and second, observer design for distributed nonlinear systems represented by TS fuzzy models. For stability analysis, our basic assumption is that a stable subsystem exists. To this system, new subsystems are added online, one-by-one, so that the distributed system grows in time. Each subsystem is represented by a TS fuzzy model and each individual subsystem (i.e., without the interconnection terms) is stable. This assumption is commonly adopted in the literature. The coupling between the subsystems is realized through their states. These assumptions are valid for several distributed systems, e.g., traffic networks, electrical networks, water networks. For such systems, a centralized re-analysis or imposing certain constraints from the very first moment on the number of subsystems to be added is impracticable. The new subsystem and the actual influence of the interconnection terms due to the addition of a new subsystem only becomes known when the subsystem is actually added. Therefore, in our approach, the stability analysis is performed sequentially, as the subsystems are added, and we derive conditions that should be satisfied by the newly added subsystem or the interconnection terms in order for the whole system to be stable.

Second, the approach is then extended to observer design. We assume that a fuzzy observer is already designed for an existing subsystem or collection of subsystems. When a new subsystem is added, together with the interconnection terms, which may affect the states and/or measurements, a new observer is designed for this subsystem only, such that this new observer, combined with the existing ones, guarantees the convergence of the estimation error for the whole system to zero.

The structure of the paper is as follows. Section 2 reviews some results for cascaded fuzzy systems, that are used as starting point for the results presented in this paper. Section 3 proposes the sequential stability conditions for distributed TS fuzzy systems. The proposed observer design for is presented in Section 4. Examples are given in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

We consider a distributed system, composed of a number of subsystems, and each subsystem being represented by a TS fuzzy model. The subsystems are coupled through their states and/or measurements, i.e., the states of one subsystem may influence the dynamics and/or measurements of other subsystems.

For the ease of the notation and without loss of generality, only two subsystems are considered here. Note, however, that the procedure can be applied sequentially, if more subsystems are added.

In this paper, we address in Section 3 the stability analysis of autonomous fuzzy system expressed as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z}) A_i \boldsymbol{x} \tag{1}$$

where $A_i, i = 1, 2, \ldots, m$ represent the local linear models, $w_i(\boldsymbol{z})$ is the corresponding normalized membership function, and $\boldsymbol{z}$ a vector of scheduling variables, that may depend on the inputs, outputs, states of the system, or other (measured) exogenous variables. We also consider in Section 4 the design of observers of the form

$$\begin{aligned}
\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{z}})(A_i \widehat{\boldsymbol{x}} + B_i \boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\
\widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{z}}) C_i \widehat{\boldsymbol{x}}
\end{aligned} \tag{2}$$

for fuzzy systems

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_i \boldsymbol{x} + B_i \boldsymbol{u}) \\
\boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{z}) C_i \boldsymbol{x}
\end{aligned} \tag{3}$$

Stability and design conditions for TS fuzzy systems generally depend on the feasibility of an associated LMI problem [11,12,14,43]. These conditions are usually conservative, but the conservativeness may be reduced for cascaded systems. Our results start from existing stability conditions for cascaded TS systems, and therefore some of the relevant conditions for this class of systems are reviewed below. Throughout the paper it is assumed that the membership functions are normalized, I denotes the identity matrix of the appropriate dimension, $\mathcal{H}(A)$ denotes the Hermitian of the matrix A, i.e., $\mathcal{H}(A) = A + A^T$, $\|\cdot\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.

Cascaded TS systems represent a special case of distributed TS systems. For cascaded systems, conditions ensuring their stability and results for observer design have been reported in [44]. These results represent a starting point for the research presented in this paper and they are therefore summarized in the remainder of this section.

2.1. Stability of cascaded TS systems

System (1) is cascaded, if the system matrices of the model (1) for each rule $i = 1, 2, \ldots, m$ can be written as

$$A_i = \begin{pmatrix} A_{1i} & 0 \\ A_{21i} & A_{2i} \end{pmatrix} \tag{4}$$

For such systems, the following stability condition [45] has been formulated:

Theorem 1. *System (1), with the system matrices of the form (4) is globally exponentially stable if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$ *so that for all* $i = 1, 2, \ldots, m$, *the following LMIs hold*:

$$\begin{aligned} &\mathcal{H}(P_1 A_{1i}) < -2Q_1 \\ &\mathcal{H}(P_2 A_{2i}) < -2Q_2 \end{aligned} \tag{5}$$

Another condition that is used in this paper is one referring to systems subjected to vanishing disturbances. Consider the following perturbed fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(z) A_i \boldsymbol{x} + \boldsymbol{f}(t, \boldsymbol{x}) \tag{6}$$

and the common assumption that $\boldsymbol{f}$ is Lipschitz in $\boldsymbol{x}$, i.e., there exists $\mu > 0$ so that $\|\boldsymbol{f}(t, \boldsymbol{x})\| \leq \mu \|\boldsymbol{x}\|$, for all t and $\boldsymbol{x}$. With these assumptions, a sufficient stability condition can be formalized by the following theorem [46].

Theorem 2. *System (6) is exponentially stable if there exist matrices* $P = P^T > 0$, $Q = Q^T > 0$, *so that the following LMIs hold*:

$$\begin{aligned} &\begin{pmatrix} Q - \mu^2 & P \\ P & I \end{pmatrix} > 0 \\ &\mathcal{H}(P A_i) < -Q, \quad i = 1, 2, \ldots, m \end{aligned} \tag{7}$$

2.2. Observer design for cascaded fuzzy systems

If the system (3) is cascaded, i.e., the system matrices A_i and C_i, $i = 1, 2, \ldots, m$ are in cascaded form, observers can be designed individually for each subsystem and each rule, with the overall observer gain having the form $L_i = \begin{pmatrix} L_{1i} & 0 \\ 0 & L_{2i} \end{pmatrix}$, where i denotes the rule number. Then, the dynamics of the error $\boldsymbol{e} = \boldsymbol{x} = \hat{\boldsymbol{x}}$ can be formulated as

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m} \sum_{j=1}^{m} w_i(\hat{z}) w_j(\hat{z}) (A_i - L_i C_j) \boldsymbol{e} + \Delta \tag{8}$$

with $\Delta = \sum_{i=1}^{m}(w_i(z) - w_i(\hat{z}))(A_i\boldsymbol{x} + B_i\boldsymbol{u})$ and $A_i - L_iC_j$ also having a cascaded form. If the scheduling vector z does not depend on the states to be estimated, then $\Delta = 0$ and Theorem 1 can be applied directly. Otherwise, using Theorem 2, the following result has been formulated [44]:

Theorem 3. *The cascaded error system* (8) *is asymptotically stable, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q = Q^T > 0$, $\mu > 0$ *and two continuous functions* $\theta_1, \theta_2 : \mathcal{R}^+ \to \mathcal{R}^+$ *such that*

$$\mathcal{H}(P_1(A_{1i} - L_{1i}C_{1j})) < -Q \quad \forall i, j : \exists z : w_i(\hat{z})w_j(\hat{z}) \neq 0$$

$$\|(w_i(z) - w_i(\hat{z}))(A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u})\| \leq \mu\|\boldsymbol{e}_1\| \quad \forall z, \hat{z}$$

$$\begin{pmatrix} Q - \mu^2 & P \\ P & I \end{pmatrix} > 0$$

$$\mathcal{H}(P_2(A_{2i} - L_{2i}C_{1j})) < 0 \quad \forall i, j : \exists z : w_i(\hat{z})w_j(\hat{z}) \neq 0$$

$$\|(w_i(z) - w_i(\hat{z}))(A_{21i}\boldsymbol{x}_1 + A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u})\| \leq \theta_1(\|\boldsymbol{e}_1\|) + \theta_2(\|\boldsymbol{e}_1\|)\|\boldsymbol{e}_2\|$$

Although the above conditions are not LMIs, they can easily be formulated as LMIs, using the change of variables $M_i = P^{-1}L_i$, $i = 1, 2, \ldots, m$.

Note that Theorems 1 and 3 are valid only for cascaded systems. In what follows, we use these theorems as a starting point for distributed systems, i.e., systems in which the influence between the subsystems is in both directions.

3. Sequential stability analysis of coupled fuzzy systems

In this section, we propose conditions to establish the stability of a TS system as subsystems are added sequentially to it. We also formulate these conditions as an LMI problem, which is easy to solve.

Consider a distributed system, with each subsystem being represented by a TS fuzzy model, where the influence of the subsystems is in both directions, i.e., a subsystem influences other subsystems and vice-versa, it is influenced by other subsystems. The subsystems are coupled through their states. The structure of the system is not fixed, i.e., new subsystems can be added online. In such a case, a centralized re-analysis of the stability of the whole system each time a new subsystem is added or removed, in general involves large computational costs and may easily become intractable. Therefore, we consider sequential analysis, based on the (already established) stability of the existing system, on the newly added subsystem, and on the interconnection terms introduced by the new subsystem. For the ease of notation and without loss of generality, only two subsystems are considered in this paper. However, the procedure can be applied sequentially for more subsystems.

In order to illustrate the main idea of our approach, consider the following example.

Example 1. Consider a TS system consisting of two subsystems

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{2} w_i(z)A_i\boldsymbol{x}$$

with the local matrices given as $A_i = \begin{pmatrix} A_{1i} & A_{12i} \\ A_{21i} & A_{2i} \end{pmatrix}$, for rule i. With no assumption on the membership functions (except that they are normalized), one can use a common Lyapunov matrix $\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}$, leading to the well-known conditions for stability

$$\begin{pmatrix} P_1A_{1i} + A_{1i}^TP_1 & P_1A_{12i} + A_{21i}^TP_2 \\ P_2A_{21i} + A_{12i}^TP_1 & P_2A_{2i} + A_{2i}^TP_2 \end{pmatrix} < 0 \tag{9}$$

In short, let us denote the above inequality by $\begin{pmatrix} R_{1i} & M_{1i} \\ M_{1i}^T & R_{2i} \end{pmatrix} < 0$. If at least bounds on the interconnection terms A_{12i}, A_{21i} are known, and we know that no further subsystems will be added to this system, then the approaches from the

Refs. [28,29,33] are suitable. Note that the common consequence of these approaches is that each subsystem is robustly stable so that the "perturbation" of the other subsystems does not influence its stability.

Assume now that the subsystems are added online, but it is not known how many subsystems will be added or how strong the introduced interconnection terms will be. Suppose that at a certain moment only the subsystem with matrices A_{2i} exists, and we know that it is stable, i.e., there exists P_2 such that $R_{2i} < R_2 < 0$, with R_2 a symmetric matrix. The subsystem with matrices A_{1i} and interconnection terms A_{12i} and A_{21i} is added online, with no prior knowledge on these terms.

The basis of our approach is to use the Schur complement, and rewrite (9) as

$$R_2 < 0$$

$$R_{1i} - M_{1i}^T R_2^{-1} M_{1i} < 0$$

and solve it in two steps, thereby establishing sequentially the stability of the system. □

Consider now a more general TS system. Assume that there exists a subsystem described as

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m_2'} w_{2i}'(z_2')(A_{2i}'\boldsymbol{x}_2) \tag{10}$$

At a certain moment in time, another subsystem is connected to this system, with the dynamics given by

$$\dot{\boldsymbol{x}}_1 = \sum_{i=1}^{m_1} w_{1i}(z_1)(A_{1i}\boldsymbol{x}_1 + A_{12i}\boldsymbol{x}_2) \tag{11}$$

Due to the connection from this new subsystem to the existing one, the dynamics of the model of the existing subsystem change to

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m_2} w_{2i}(z_2)(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1) \tag{12}$$

i.e., in general both the membership functions and the local matrices change. In this paper, we only consider the case when the membership functions change, assuming that the local matrices remain the same, i.e., $\{A_{2i}' | i = 1, 2, \ldots, m_2'\} \equiv \{A_{2i} | i = 1, 2, \ldots, m_2\}$. Such an assumption holds for distributed systems in which the addition of a new subsystem does not influence the individual dynamics of the existing subsystems, but instead can be considered as a new input acting on the system. Distributed systems for which this assumption holds are, e.g., material flow systems, traffic networks, water logistics, production systems, etc.

The assumption can be formulated as follows:

Assumption 1. The local state matrices of the existing subsystem do not change by the addition of the new subsystem.

Note that the interconnection terms are not known before the new subsystem is added, nor are the local matrices of the new subsystem. The restrictiveness of this assumption largely depends on how the fuzzy model is obtained from a nonlinear model. For instance, consider the original system (10). If, after adding the new subsystem, its dynamics change to

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m'} w_i'(z')(A_{2i}\boldsymbol{x}_2) + \mathbf{A}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_1$$

with $\mathbf{A}$ a smooth nonlinear matrix function that may depend on both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, then using the sector nonlinearity approach, the local models of the original subsystem will remain the same (in fact they are repeated in several rules), although the membership functions will change. The interconnection term being $\mathbf{A}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_1$, using the sector nonlinearity approach, it can be exactly represented as $\mathbf{A}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_1 = \sum_{i=1}^{r} h_i(z_n)(A_{21i}\boldsymbol{x}_1)$, with z_n depending on $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and h_i being normalized membership functions. Exploiting the fact that the membership functions w_i' are also

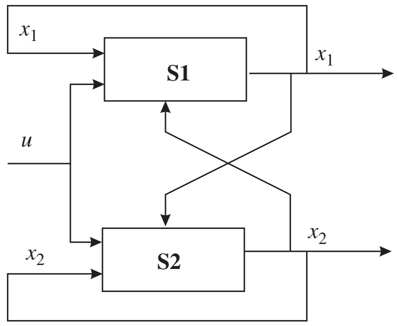

Fig. 1. Two subsystems coupled through their states.

normalized, one has

$$\begin{aligned}\dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m'} w'_i(z')(A_{2i}\boldsymbol{x}_2) + \mathbf{A}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_1 \\ &= \sum_{i=1}^{m'} w'_i(z')(A_{2i}\boldsymbol{x}_2) + \sum_{i=1}^{r} h_i(z_n)(A_{21i}\boldsymbol{x}_1) \\ &= \sum_{i=1}^{m'} w'_i(z') \sum_{j=1}^{r} h_j(z_n)(A_{2i}\boldsymbol{x}_2 + A_{21j}\boldsymbol{x}_1) \\ &= \sum_{i=1}^{m'} \sum_{j=1}^{r} w'_i(z')h_j(z_n)(A_{2i}\boldsymbol{x}_2 + A_{21j}\boldsymbol{x}_1) \\ &= \sum_{i=1}^{m} w_i(z)(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1)\end{aligned}$$

When the new subsystem is added, and Assumption 1 is satisfied, the whole system, i.e., the subsystem added (with states $\boldsymbol{x}_1$), the existing subsystem (with states $\boldsymbol{x}_2$) and the interconnection terms are expressed together as

$$\begin{aligned}\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_i(z)(A_{1i}\boldsymbol{x}_1 + A_{12i}\boldsymbol{x}_2) \\ \dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_i(z)(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1)\end{aligned} \tag{13}$$

The structure of system (13) is presented in Fig. 1.

For such a system, we have formulated the following stability conditions [47]:

Theorem 4. *The system* (13) *is asymptotically stable, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, *so that*

$$\begin{aligned}&\mathcal{H}(P_1 A_{1i}) < -2Q_1, \quad i = 1, 2, \ldots, m \\ &\mathcal{H}(P_2 A_{2i}) < -2Q_2, \quad i = 1, 2, \ldots, m \\ &\lambda_{\min}(Q_1) \geq \max_i \|P_1 A_{12i}\| \\ &\frac{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)}\end{aligned}$$

where $\lambda_{\min}(\cdot)$ *is the eigenvalue with the smallest absolute magnitude.*

For the completeness of the paper and since results in the following sections make use of steps of the proof of this theorem, we also repeat here the proof.

Proof. Consider first the following part of the system (13):

$$\dot{\boldsymbol{x}}_1 = \sum_{i=1}^{m} w_i(z)(A_{1i}\boldsymbol{x}_1)$$
$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m} w_i(z)(A_{2i}\boldsymbol{x}_2 + A_{21i}\boldsymbol{x}_1) \tag{14}$$

This is a cascaded system. According to Theorem 1, this system is exponentially stable, if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$ so that

$$\mathcal{H}(P_1 A_{1i}) < -2Q_1, \quad i = 1, 2, \ldots, m$$
$$\mathcal{H}(P_2 A_{2i}) < -2Q_2, \quad i = 1, 2, \ldots, m \tag{15}$$

In order to make the step from the stable cascaded system to the analysis of the distributed system, a Lyapunov function is needed. One way of constructing the Lyapunov function using P_1 and P_2 is by considering the function $V_c = \boldsymbol{x}^T \operatorname{diag}(\alpha P_1, P_2)\boldsymbol{x}$. The advantage of this choice is that it allows one to determine $\alpha \in \mathcal{R}^+$ so that $\dot{V}_c < -2\boldsymbol{x}^T Q\boldsymbol{x}$, with $Q = \operatorname{diag}(\alpha Q_1, Q_2)$:

$$\dot{V}_c = \sum_{i=1}^{m} w_i(z)\boldsymbol{x}^T \begin{pmatrix} \alpha\mathcal{H}(P_1 A_{1i}) & A_{21i}^T P_2 \\ P_2 A_{21i} & \mathcal{H}(P_2 A_{2i}) \end{pmatrix} \boldsymbol{x}$$

Then, $\dot{V}_c < -2\boldsymbol{x}^T Q\boldsymbol{x}$, if

$$\begin{pmatrix} \alpha\mathcal{H}(P_1 A_{1i}) & A_{21}{}_i^T P_2 \\ P_2 A_{21i} & \mathcal{H}(P_2 A_{2i}) \end{pmatrix} < -2 \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

or

$$\begin{pmatrix} \alpha\mathcal{H}(P_1 A_{1i} + Q_1) & A_{21}{}_i^T P_2 \\ P_2 A_{21i} & \mathcal{H}(P_2 A_{2i} + Q_2) \end{pmatrix} < 0$$

Using the Schur complement, we have

$$\alpha\mathcal{H}(P_1 A_{1i} + Q_1) - (A_{21i}^T P_2)(\mathcal{H}(P_2 A_{2i} + Q_2))^{-1} P_2 A_{21i} < 0$$

which is true if α is chosen such that

$$\alpha > \frac{1}{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))} \cdot \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2))} \tag{16}$$

where $\lambda_{\min}(\cdot)$ denotes the eigenvalue with the smallest absolute magnitude. Now, consider the full system (13). By using the above constructed V_c as a candidate Lyapunov function for (13), we obtain

$$\dot{V}_c = \sum_{i=1}^{m} w_i(z)\boldsymbol{x}^T \left[\begin{pmatrix} \alpha\mathcal{H}(P_1 A_{1i}) & A_{21i}^T P_2 \\ P_2 A_{21i} & \mathcal{H}(P_2 A_{2i}) \end{pmatrix} + \begin{pmatrix} 0 & \alpha P_1 A_{12i} \\ \alpha A_{12i}^T P_1 & 0 \end{pmatrix} \right] \boldsymbol{x}$$
$$< -2\boldsymbol{x}^T \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \boldsymbol{x} + 2\boldsymbol{x}^T \alpha \max_i \| P_1 A_{12i} \| I\boldsymbol{x}$$
$$< -2\boldsymbol{x}^T \begin{pmatrix} \alpha(Q_1 - \max_i \|P_1 A_{12i}\| I) & 0 \\ 0 & Q_2 - \alpha \max_i \|P_1 A_{12i}\| I \end{pmatrix} \boldsymbol{x}$$

which leads to the conditions

$$\lambda_{\min}(Q_1) > \max_i \|P_1 A_{12i}\| \tag{17}$$

$$\lambda_{\min}(Q_2) > \alpha \max_i \|P_1 A_{12i}\| \tag{18}$$

Combining (16) and (18), we find a α exists, and V_c is a Lyapunov function for the whole system if

$$\frac{\lambda_{\min}(Q_2)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2))}$$

or

$$\frac{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2))} \qquad \square$$

Remark. If $A_{12i} = 0$, for all $i = 1, 2, \ldots, m$ or $A_{21i} = 0$, for all $i = 1, 2, \ldots, m$, then based on Theorem 1, the system (13) is stable if the individual subsystems are stable, and the last two conditions are not required.

Note that the conditions of Theorem 4 are not LMIs. In order to solve them we consider a two-step procedure presented in the following algorithm, which allows the conditions to be formulated as LMIs:

Algorithm 1.

1. The existing system

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m} w_i(z) A_{2i} \boldsymbol{x}$$

is already proven to be stable using a quadratic Lyapunov function and therefore P_2 and Q_2 such that $\mathcal{H}(P_2 A_{2i}) < -2Q_2$ have been computed. Thanks to this, when adding the new subsystem, with the interconnection terms, the value of

$$\gamma = \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2))}$$

can be computed.

2. Now, for the added subsystem and the corresponding interconnection terms we have the conditions:

$$\mathcal{H}(P_1 A_{1i}) < -2Q_1, \quad i = 1, 2, \ldots, m$$

$$\lambda_{\min}(Q_1) \geq \max_i \|P_1 A_{12i}\|$$

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1)) > \gamma \max_i \|P_1 A_{12i}\|$$

which are satisfied if the LMIs

$$\mathcal{H}(P_1 A_{1i} + Q_1) < -2t_1 I, \quad i = 1, 2, \ldots, m$$

$$Q_1 > t_2 I$$

$$\begin{pmatrix} t_2 I & \max_i \|A_{12i}\| P_1 \\ \max_i \|A_{12i}\| P_1 & t_2 I \end{pmatrix} > 0 \tag{19}$$

$$\begin{pmatrix} t_1 I & \gamma \max_i \|A_{12i}\| P_1 \\ \gamma \max_i \|A_{12i}\| P_1 & t_1 I \end{pmatrix} > 0$$

are feasible.

Moreover, if one takes into consideration that new subsystems will be added to the whole system (13), the analysis of the new subsystems can be facilitated by minimizing the expression:

$$\frac{\|P_1\|^2}{\lambda_{\min}(Q_1)\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))}$$

which will in turn minimize the bound γ computed for the system (13).

This can be achieved by solving the LMI-based convex problem: *find* $P_1 = P_1^T > 0$, $Q_1 = Q_1^T > 0$, *and maximize* t_1, t_2, t_3 *subject to* (19) *and*

$$P_1 < t_3 I$$

A shortcoming of the approach at this point is that although the stability analysis of the second subsystem has been performed, and $V_c = \begin{pmatrix} \alpha P_1 & 0 \\ 0 & P_2 \end{pmatrix}$ is used as a Lyapunov function, all that is known is that $\dot{V}_c < 0$. With a similar reasoning, when the next subsystem is added, it is required that $\dot{V}_c \leq -2\boldsymbol{x}^T Q \boldsymbol{x}$, for some $Q = Q^T > 0$. To obtain such a Q, consider the derivative of $\dot{V}_c$. We have

$$\dot{V}_c < -2\boldsymbol{x}^T \begin{pmatrix} \alpha(Q_1 - \max_i \|P_1 A_{12i}\| I) & 0 \\ 0 & Q_2 - \alpha \max_i \|P_1 A_{12i}\| I \end{pmatrix} \boldsymbol{x}$$

By imposing that

$$\begin{pmatrix} \alpha(Q_1 - \max_i \|P_1 A_{12i}\| I) & 0 \\ 0 & Q_2 - \alpha \max_i \|P_1 A_{12i}\| I \end{pmatrix} > \beta \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

for some arbitrary $\beta \in (0, 1)$, the following conditions are obtained:

$$Q_1 - \max_i \|P_1 A_{12i}\| I > \beta Q_1$$

$$Q_2 - \alpha \max_i \|P_1 A_{12i}\| I > \beta Q_2$$

i.e.,

$$\begin{aligned} &(1-\beta) Q_1 > \max_i \|P_1 A_{12i}\| I \\ &(1-\beta) Q_2 > \alpha \max_i \|P_1 A_{12i}\| I \end{aligned} \tag{20}$$

Combining (20) and the conditions of Theorem 4, we obtain that

Corollary 1. $V = \boldsymbol{x}^T \begin{pmatrix} \alpha P_1 & 0 \\ 0 & P_2 \end{pmatrix} \boldsymbol{x}$ *is a Lyapunov function for* (13) *and* $\dot{V} < -2\boldsymbol{x}^T \beta \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \boldsymbol{x}$ *for an arbitrary* $\beta \in (0, 1)$ *if*

$$\mathcal{H}(P_1 A_{1i}) < -2Q_1, \quad i = 1, 2, \ldots, m$$

$$\mathcal{H}(P_2 A_{2i}) < -2Q_2, \quad i = 1, 2, \ldots, m$$

$$(1-\beta)\lambda_{\min}(Q_1) \geq \max_i \|P_1 A_{12i}\| \tag{21}$$

$$\frac{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1)}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|A_{21i}^T P_2\|^2}{(1-\beta)\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)} \qquad \square$$

Recall, that we assumed that the interconnection terms or bounds on them are not known before adding a new subsystem. However, if $c_k = \max_{ij} \|A_{kij}\|$, i.e., a bound on the interconnection terms is known beforehand, the analysis of the subsystems can be decoupled and the following result can be stated:

Theorem 5. *Given* $c_1 = \max_i \|A_{12i}\|$ *and* $c_2 = \max_i \|A_{21i}\|$, *the distributed system* (13) *is exponentially stable, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, *so that*

$$\mathcal{H}(P_1 A_{1i}) < -2Q_1, \quad i = 1, 2, \ldots, m$$

$$\mathcal{H}(P_2 A_{2i}) < -2Q_2, \quad i = 1, 2, \ldots, m$$

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1)) > \lambda_{\min}(Q_1)$$

$$\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)) > \lambda_{\min}(Q_2)$$

$$\lambda_{\min}(Q_1) \geq c_1 \|P_1\|$$

$$\lambda_{\min}(Q_2) \geq c_2 \|P_2\|$$

where $\lambda_{\min}(\cdot)$ *is the eigenvalue with the smallest absolute magnitude.*

Proof. Consider the last condition of Theorem 4:

$$\frac{\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))}{\max_i \|P_1 A_{12i}\|} > \frac{\max_i \|A_{21i}^T P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2))}, \text{ i.e.,}$$

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)) > \max_i \|P_1 A_{12i}\| \max_i \|A_{21i}^T P_2\|^2$$

The third condition of Theorem 4 already states that

$$\lambda_{\min}(Q_1) \geq \max_i \|P_1 A_{12i}\| \tag{22}$$

If Q_2 is similarly restricted, i.e., the condition

$$\lambda_{\min}(Q_2) \geq \max_i \|P_2 A_{21i}\| \tag{23}$$

is imposed, then the last condition of Theorem 4 becomes

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)) > \lambda_{\min}(Q_1)\lambda_{\min}^2(Q_2)$$

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1))\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)) > \lambda_{\min}(Q_1)\lambda_{\min}(Q_2)$$

which is satisfied if

$$\lambda_{\min}(\mathcal{H}(P_1 A_{1i} + Q_1)) > \lambda_{\min}(Q_1)$$

$$\lambda_{\min}(\mathcal{H}(P_2 A_{2i} + Q_2)) > \lambda_{\min}(Q_2)$$

However, since only the bounds on the interconnection terms c_1 and c_2 are known, instead of (22) and (23) we have to use

$$\lambda_{\min}(Q_1) \geq c_1 \|P_1\|$$

$$\lambda_{\min}(Q_2) \geq c_2 \|P_2\| \tag{24}$$

Together with the restrictions on Q_1 and Q_2, and (24), we obtain the conditions expressed in Theorem 5. □

Note that the conditions of Theorem 5 are similar to those reported in [28,29]. The decoupled analysis has the advantages that (1) the analysis of the subsystems can be performed in parallel and (2) each subsystem has to dominate one interconnection term, whose approach is less conservative when the strength of the interconnection terms is approximately the same and both are weak. However, this result can only be obtained if bounds on the interconnection terms that are introduced to the system by the addition of a new subsystem are known beforehand. This condition is not needed for Theorem 4, as, thanks to the sequential analysis, the interconnection terms only need to be known when the subsystem that introduces them is analyzed.

Theorem 5, similar to current results for stability analysis and stabilization of fuzzy large scale systems [26,28,29, 43,33] is comparable to perturbation methods with weak coupling (see [15] and the references therein). In fact, the assumption that the coupling is "weak enough", compared to the dynamics of the individual subsystems is necessary for the controller (or the analysis) to be decoupled.

In contrast, Theorem 4 and the resulting Algorithm 1 are comparable to methods developed for strong coupling, i.e., only one of the subsystems has to converge quickly enough so that stability is preserved. This approach can also be thought of as an asymmetrical weak coupling, i.e., only one of the influences has to be weak enough for stability to be preserved.

4. Sequential observer design

This section extends the sequential approach to observer design for TS fuzzy systems.

4.1. Preliminaries

Again, consider a distributed system where each subsystem is represented by a TS fuzzy model. New subsystems will be added online, one-by-one, and our goal is to design an asymptotically stable observer for the whole system. Since the subsystems are added one-by-one, we consider sequential design, where an observer is designed whenever a new subsystem is added, in such a way that the overall observer is stable, but without modifying the already existing observers.

In this paper we assume that the estimated states of the subsystems are available to all other subsystems that are interconnected with it. However, we do not assume that the subsystems are stabilized.

For the ease of notation and without loss of generality, only two subsystems are considered. The observer structure is depicted in Fig. 2.

The fuzzy system considered consists of two subsystems:

$$
\begin{aligned}
\dot{\boldsymbol{x}}_1 &= \sum_{i=1}^{m} w_i(z)(A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_2)\\
\boldsymbol{y}_1 &= \sum_{i=1}^{m} w_i(z)(C_{11i}\boldsymbol{x}_1 + C_{12i}\boldsymbol{x}_2)\\
\dot{\boldsymbol{x}}_2 &= \sum_{i=1}^{m} w_i(z)(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1)\\
\boldsymbol{y}_2 &= \sum_{i=1}^{m} w_i(z)(C_{22i}\boldsymbol{x}_2 + C_{21i}\boldsymbol{x}_1)
\end{aligned}
\tag{25}
$$

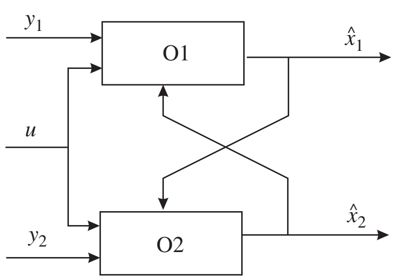

Fig. 2. Distributed observer for two subsystems.

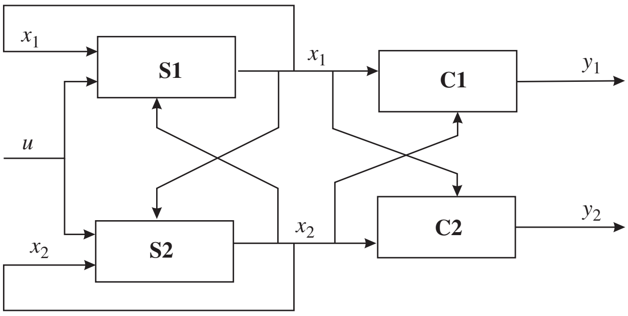

Fig. 3. Two subsystems coupled through their states and measurements.

and the observer is of the form:

$$\dot{\hat{x}}_1 = \sum_{i=1}^{m} w_i(\hat{z})(A_{1i}\hat{x}_1 + B_{1i}u + A_{12i}\hat{x}_2 + L_{1i}(y_1 - \hat{y}_1))$$

$$\hat{y}_1 = \sum_{i=1}^{m} w_i(\hat{z})(C_{11i}\hat{x}_1 + C_{12i}\hat{x}_2)$$

$$\dot{\hat{x}}_2 = \sum_{i=1}^{m} w_i(\hat{z})(A_{2i}\hat{x}_2 + B_{2i}u + A_{21i}\hat{x}_1 + L_{2i}(y_2 - \hat{y}_2))$$

$$\hat{y}_2 = \sum_{i=1}^{m} w_i(\hat{z})(C_{22i}\hat{x}_2 + C_{21i}\hat{x}_1) \quad (26)$$

The goal is to design the observer gains L_{1i}, $i = 1, 2, \ldots, m$ for each rule of the subsystem with states x_1 so that (26) is a stable observer, given that the gains L_{2i}, $i = 1, 2, \ldots, m$ have already been designed such that the observer

$$\dot{\hat{x}}_2 = \sum_{i=1}^{m} w_i(\hat{z})(A_{2i}\hat{x}_2 + B_{2i}u + L_{2i}(y_2 - \hat{y}_2))$$

$$\hat{y}_2 = \sum_{i=1}^{m} w_i(\hat{z})C_{22i}\hat{x}_2$$

is stable for the subsystem without the interconnection terms:

$$\dot{x}_2 = \sum_{i=1}^{m} w_i(z)(A_{2i}x_2 + B_{2i}u)$$

$$y_2 = \sum_{i=1}^{m} w_i(z)C_{22i}x_2$$

The system structure considered exhibits coupling in both states and measurements. Such a system is presented in Fig. 3. Two cases are distinguished, according to whether or not the scheduling vector depends on some of the states to be estimated.

4.2. State-independent scheduling vector

In this section, we consider the case when the scheduling vector does not depend on states to be estimated. For this case, we first propose non-LMI conditions for the stability of the dynamics of the global estimation error, from which we derive LMI conditions. We finally show that if a bound on the interconnection terms is known beforehand, a decoupled observer design can be performed, similar to those in the literature.

4.2.1. Distributed observer design

If the scheduling vector does not depend on the states to be estimated, the error systems can be expressed as

$$\begin{aligned}\dot{\boldsymbol{e}}_1 &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(z)w_j(z)[A_{1i}\boldsymbol{e}_1 + A_{12i}\boldsymbol{e}_2 - L_{1i}C_{1j}\boldsymbol{e}]\\ \boldsymbol{e}_{y1} &= \sum_{i=1}^{m} w_i(z)C_{1i}\boldsymbol{e}\\ \dot{\boldsymbol{e}}_2 &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(z)w_j(z)[A_{2i}\boldsymbol{e}_2 + A_{21i}\boldsymbol{e}_1 - L_{2i}C_{2j}\boldsymbol{e}]\\ \boldsymbol{e}_{y2} &= \sum_{i=1}^{m} w_i(z)C_{2i}\boldsymbol{e}\end{aligned} \tag{27}$$

where $C_{1i} = [C_{11i}\ C_{12i}]$ and $C_{2i} = [C_{21i}\ C_{22i}]$, or

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(z)w_j(z)\begin{pmatrix} A_{1i} - L_{1i}C_{11j} & A_{12i} - L_{1i}C_{12j}\\ A_{21i} - L_{2i}C_{21j} & A_{2i} - L_{2i}C_{22j}\end{pmatrix}\boldsymbol{e} \tag{28}$$

Note that since L_{1i}, $i = 1, 2, \ldots, m$ have to be designed, a simple special case is when there exist $P_1 = P_1^T > 0$ and L_{1i}, so that $\mathcal{H}(P_1(A_{1i} - L_{1i}C_{11j})) < 0$ and $A_{12i} - L_{1i}C_{12j} = 0\ \forall i, j : \exists z : w_i(z)w_j(z) \neq 0$. In this case the error system (28) is cascaded, and further restrictions are not necessary, and the stability conditions can be summarized as follows.

Corollary 2. *The error system* (28) *is asymptotically stable if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, L_{1i}, L_{2i}, $i = 1, 2, \ldots, m$ *so that* $\forall i, j : \exists z : w_i(z)w_j(z) \neq 0$

$$\begin{aligned}&\mathcal{H}(P_1(A_{1i} - L_{1i}C_{11j})) < 0\\ &\mathcal{H}(P_2(A_{2i} - L_{2i}C_{22j})) < 0\\ &A_{12i} - L_{1i}C_{12j} = 0\end{aligned}$$

Proof. This follows directly from Theorem 3, applied for the case when $\hat{z} = z$. □

Note that the third condition of Corollary 2 is more likely to be satisfied if the measurement matrix is common for the rules of a subsystem and the coupling is present only in measurements. However, in general this is not the case and it is not possible to find such L_{1i}. Therefore, the results from Section 3 can be appropriately modified:

Corollary 3. *The error system* (28) *is exponentially stable, if there exist* L_{1i}, L_{2i}, $i = 1, 2, \ldots, m$, $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, *so that*

$$\begin{aligned}&\mathcal{H}(P_1G_{1ij}) < -2Q_1 \quad \forall i, j : \exists z : w_i(z)w_j(z) \neq 0\\ &\mathcal{H}(P_2G_{2ij}) < -2Q_2 \quad \forall i, j : \exists z : w_i(z)w_j(z) \neq 0\\ &\lambda_{\min}(Q_1) \geq \max_{ij}\|P_1G_{12ij}\|\\ &\frac{\lambda_{\min}(\mathcal{H}(P_1G_{1ij} + Q_1))}{\max_{ij}\|P_1G_{12ij}\|} > \frac{\max_{ij}\|P_2G_{21ij}\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2G_{2ij} + Q_2))}\end{aligned}$$

where $G_{1ij} = A_{1i} - L_{1i}C_{11j}$, $G_{2ij} = A_{2i} - L_{2i}C_{22j}$, $G_{12ij} = A_{12i} - L_{1i}C_{12j}$, *and* $G_{21ij} = A_{21i} - L_{2i}C_{21j}$.

Proof. This follows directly from Theorem 4 applied to the error dynamics (28). □

Remark. In order to facilitate the design of observer for the next subsystem, the conditions of Corollary 1 can be appropriately modified.

4.2.2. LMI conditions

Note that Corollary 3 leads to a sequential implementation, similar to Algorithm 1. Once a stable observer is designed for the subsystem

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u})$$

$$\boldsymbol{y}_2 = \sum_{i=1}^{m} w_i(\boldsymbol{z})C_{22i}\boldsymbol{x}_2$$

the matrices P_2, Q_2, and the gains L_{2i}, $i = 1, 2, \ldots, m$ are known, and therefore, G_{2ij} can be computed. After adding the interconnection terms, G_{21ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$ also the ratio

$$\gamma = \frac{\max_{ij} \|P_2 G_{21ij}\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2 G_{2ij} + Q_2))}$$

can be computed. The conditions of Corollary 3 are then reduced to: find L_{2i}, $i = 1, 2, \ldots, m$, $P_1 = P_1^T > 0$, $Q_1 = Q_1^T > 0$, so that for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$

$$\mathcal{H}(P_1 G_{1ij}) < -2Q_1$$

$$\lambda_{\min}(Q_1) \geq \max_{ij} \|P_1 G_{12ij}\|$$

$$\lambda_{\min}(\mathcal{H}(P_1 G_{1ij} + Q_1)) > \gamma \max_{ij} \|P_1 G_{12ij}\|$$

which is satisfied if

$$\mathcal{H}(P_1 G_{1ij} + Q_1) < 0$$

$$Q_1 \geq \max_{ij} \|P_1 G_{12ij}\| I$$

$$\mathcal{H}(P_1 G_{1ij} + Q_1) < -\gamma \max_{ij} \|P_1 G_{12ij}\| I$$

The above conditions are satisfied if the following LMI is feasible, with the change of variables $M_i = P_1 L_{1i}$: find L_{2i}, $i = 1, 2, \ldots, m$, $P_1 = P_1^T > 0$, $Q_1 = Q_1^T > 0$, $t_1 > 0$, $t_2 > 0$, M_i, $i = 1, 2, \ldots, m$, so that for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$

$$\mathcal{H}(P_1 A_{1i} - M_i C_{1i} + Q_1) < -t_1 I$$

$$Q_1 > t_2 I$$

$$\begin{pmatrix} t_2 I & P_1 A_{12i} - M_i C_{21j} \\ (P_1 A_{12i} - M_i C_{21j})^T & t_2 I \end{pmatrix} > 0$$

$$\begin{pmatrix} t_1 I & P_1 \gamma A_{12i} - M_i \gamma C_{21j} \\ (P_1 \gamma A_{12i} - M_i \gamma C_{21j})^T & t_1 I \end{pmatrix} > 0$$

The observer design for the newly added subsystem can be summarized as follows.

Algorithm 2.

1. For the existing observer of the subsystem

$$\dot{\boldsymbol{x}}_2 = \sum_{i=1}^{m} w_i(\boldsymbol{z})(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u})$$

$$\boldsymbol{y}_2 = \sum_{i=1}^{m} w_i(\boldsymbol{z})C_{22i}\boldsymbol{x}_2$$

compute

$$\bar{\gamma} = \frac{\|P_2\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2G_{2ij} + Q_2))}$$

2. When the new subsystem and corresponding interconnection terms are added, compute $\gamma = \bar{\gamma}\max_{ij}\|G_{21ij}\|^2$. To design the observer for this subsystem, solve the LMI problem: find L_{2i}, $i = 1, 2, \ldots, m$, $P_1 = P_1^T > 0$, $Q_1 = Q_1^T > 0$, $t_1 > 0$, $t_2 > 0$, M_i, $i = 1, 2, \ldots, m$, so that for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$

$$\mathcal{H}(P_1A_{1i} - M_iC_{1i} + Q_1) < -t_1I$$

$$Q_1 > t_2I$$

$$\begin{pmatrix} t_2I & P_1A_{12i} - M_iC_{21j} \\ (P_1A_{12i} - M_iC_{21j})^T & t_2I \end{pmatrix} > 0$$

$$\begin{pmatrix} t_1I & P_1\gamma A_{12i} - M_i\gamma C_{21j} \\ (P_1\gamma A_{12i} - M_i\gamma C_{21j})^T & t_1I \end{pmatrix} > 0$$

4.2.3. Decoupled observer design

Note that above algorithm is useful if no bound on the interconnection terms is known before the subsystem is added. If a bound on A_{12i}, A_{21i}, C_{21i}, C_{12i}, $i = 1, 2, \ldots, m$ is known beforehand, observers can be designed independently for the subsystems, by analyzing the last condition of Corollary 3. Although the following manipulations introduce further conservativeness, the design is decoupled, and LMI conditions are obtained. The result can be stated as follows:

Corollary 4. *The error system* (28) *is exponentially stable, if there exist* L_{ki}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, $P_k = P_k^T > 0$, $Q_k = Q_k^T > 0$ *so that* $\forall i, j : \exists \boldsymbol{z} : w_i(\boldsymbol{z})w_j(\boldsymbol{z}) \neq 0$

$$\mathcal{H}(P_kG_{kij}) < -2Q_k$$

$$\lambda_{\min}(Q_k) \geq \max_i \|P_kT_{kij}\|$$

$$\lambda_{\min}(\mathcal{H}(P_kG_{kij} + Q_k)) > \max_{ij}\|P_kT_{kij}\| \tag{29}$$

where $G_{kij} = A_{ki} - L_{ki}C_{kj}$, $T_{kij} = A_{kpi} - L_{ki}C_{kpj}$ *is the interconnection term that influences the subsystem* k *and* L_{ki}, $i = 1, 2, \ldots, m$ *are the observer gains of the* k*th subsystem.*

Proof. In order to obtain similar conditions for all the subsystems, let us impose

$$\lambda_{\min}(Q_2) \geq \max_{ij}\|P_2G_{21ij}\|$$

Then, the last condition of Corollary 3 becomes

$$\frac{\max_{ij}\|P_2G_{21ij}\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2G_{2ij} + Q_2))} \leq \frac{\max_{ij}\|P_2G_{21ij}\|}{\lambda_{\min}(\mathcal{H}(P_2G_{2ij} + Q_2))}$$

an expression that is similar to that of the reciprocal of the first part of the fourth condition of the corollary, i.e.,

$$\frac{\lambda_{\min}(\mathcal{H}(P_1 G_{1ij} + Q_1))}{\max_{ij} \|P_1 G_{12ij}\|}$$

By imposing for both subsystems

$$\frac{\lambda_{\min}(\mathcal{H}(P_k G_{kij} + Q_k))}{\max_{ij} \|P_k T_{kij}\|} > 1$$

where T_{kij} is the interconnection term influencing the subsystem k, $T_{kij} = A_{kpi} - L_{ki} C_{kpj}$, $k = 1, 2$, the conditions are decoupled. Summarizing, we have the conditions:

$$\mathcal{H}(P_k G_{kij}) < -2Q_k$$

$$\lambda_{\min}(Q_k) \geq \max_i \|P_k T_{kij}\|$$

$$\lambda_{\min}(\mathcal{H}(P_k G_{kij} + Q_k)) > \max_{ij} \|P_k T_{kij}\| \qquad \square$$

Note that the conditions of Corollary 4 are not LMIs. However, an LMI problem may be formulated, which, when satisfied, also satisfies the conditions of Corollary 4, as follows:

Theorem 6. *The error system* (28) *is exponentially stable*, *if there exist* M_{ki}, $i = 1, 2, \ldots, m$, $P_k = P_k^T > 0$, $Q_k = Q_k^T$, $t_1 > 0$, $t_2 > 0$, $t_{k\mathrm{M}} > 0$, $t_{k\mathrm{m}} > 0$, *so that* $\forall i, j : \exists z : w_i(z) w_j(z) \neq 0$

$$t_{k\mathrm{m}} I \leq Q_k \leq t_{k\mathrm{M}} I$$

$$\mathcal{H}(P_k G_{kij} + Q_k) < -t_{k\mathrm{M}} I$$

$$t_{k\mathrm{m}} I \geq Q_{kA} + Q_{kC}$$

$$Q_{kA} \geq t_1 I$$

$$Q_{kC} \geq t_2 I$$

$$\begin{pmatrix} t_1 I & \mu_{Ak} P_k \\ \mu_{Ak} P_k & t_1 I \end{pmatrix} \geq 0$$

$$\begin{pmatrix} t_2 I & \mu_{Ck} M_{ki} \\ \mu_{Ck} M_{ki}^T & t_2 I \end{pmatrix} \geq 0 \tag{30}$$

Proof. Let

$$\lambda_{\min}(\mathcal{H}(P_k G_{kij} + Q_k)) > \lambda_{\min}(Q_k)$$

and $t_{k\mathrm{m}} I \leq Q_k \leq t_{k\mathrm{M}} I$. Then, the conditions (29) are satisfied if

$$t_{k\mathrm{m}} I \leq Q_k \leq t_{k\mathrm{M}} I$$

$$\mathcal{H}(P_k G_{kij} + Q_k) < -t_{k\mathrm{M}} I$$

$$t_{k\mathrm{m}} I \geq \max_{ij} \|P_k T_{kij}\| \tag{31}$$

Recall that the interconnection term T_{kij} is in fact $T_{kij} = A_{kpi} - L_{ki} C_{kpj}$, i.e., the interconnection term in the error dynamics. However, only the bounds on the interconnection terms in the subsystems are known, i.e., $\mu_{Ak} = \max_{pi} \|A_{kpi}\|$ and $\mu_{Ck} = \max_{pi} \|C_{kpi}\|$, where k is the index of the current subsystem, $k = 1, 2$. Therefore, let Q_k be the sum of two positive definite matrices, $Q_k = Q_{kA} + Q_{kC}$, which satisfy

$$Q_{kA} \geq \mu_{Ak} \|P_k\| I$$

$$Q_{kC} \geq \mu_{Ck} \max_i \|P_k L_{ki}\| I$$

The conditions above may be expressed as LMIs:

$$Q_{kA} \geq t_1 I$$
$$Q_{kC} \geq t_2 I$$
$$\begin{pmatrix} t_1 I & \mu_{Ak} P_k \\ \mu_{Ak} P_k & t_1 I \end{pmatrix} \geq 0$$
$$\begin{pmatrix} t_2 I & \mu_{Ck} M_{ki} \\ \mu_{Ck} M_{ki}^T & t_2 I \end{pmatrix} \geq 0$$

where $M_{ki} = P_k L_{ki}$.

Summarizing all the conditions, we obtain those of Theorem 6. These conditions are not only decoupled, but also expressed as LMIs. □

Note however, that this result can only be used if a bound on the possible interconnection term is known. Also, due to the restrictions imposed while deriving LMIs, the conditions of Theorem 6 are more conservative than those of Corollary 3.

4.3. State-dependent scheduling vector

Consider now the case when the scheduling vector depends on the states to be estimated. For the simplicity of the notation, only the case when the measurement matrices are common for all the rules of a subsystem is presented. Note however, that if the measurement matrices are different for each rule, the observers can be designed in a similar fashion.

The error system (similar to Section 4.2) can be expressed as

$$\begin{aligned}
\dot{\boldsymbol{e}}_1 &= \sum_{i=1}^{m} w_i(\widehat{z})[A_{1i}\boldsymbol{e}_1 + A_{12i}\boldsymbol{e}_2 - L_{1i}C_1\boldsymbol{e}] + \sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z}))(A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_2) \\
\boldsymbol{e}_{y1} &= C_1\boldsymbol{e} \\
\dot{\boldsymbol{e}}_2 &= \sum_{i=1}^{m} w_i(\widehat{z})[A_{2i}\boldsymbol{e}_2 + A_{21i}\boldsymbol{e}_1 - L_{2i}C_2\boldsymbol{e}] + \sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z}))(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1) \\
\boldsymbol{e}_{y2} &= C_2\boldsymbol{e}
\end{aligned} \tag{32}$$

or

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m} w_i(\widehat{z}) \begin{pmatrix} A_{1i} - L_{1i}C_{11} & A_{12i} - L_{1i}C_{12} \\ A_{21i} - L_{2i}C_{21} & A_{2i} - L_{2i}C_{22} \end{pmatrix} \boldsymbol{e} + \sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z})) \begin{pmatrix} A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_2 \\ A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1 \end{pmatrix} \tag{33}$$

In order to be able to derive LMI conditions, it is assumed that

$$\Delta = \sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z})) \begin{pmatrix} A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_2 \\ A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1 \end{pmatrix}$$

can be written as $\Delta = F\boldsymbol{e}$, with F a bounded uncertainty matrix, $\|F\| \leq \mu$. Consider now the distributed observer design. For the already existing subsystem the error is

$$\begin{aligned}
\dot{\boldsymbol{e}}_2 &= \sum_{i=1}^{m} w_i(\hat{\bar{z}})[A_{2i}\boldsymbol{e}_2 - L_{2i}C_{22}\boldsymbol{e}_2] + \sum_{i=1}^{m}(w_i(\bar{z}) - w_i(\hat{\bar{z}}))(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u}) \\
\boldsymbol{e}_{y2} &= C_{22}\boldsymbol{e}_2
\end{aligned} \tag{34}$$

where $\bar{z}$ depends *only* on the states of this subsystem. For this subsystem, there already exists a condition on the model–observer mismatch, i.e., $\|\bar{\Delta}\| = \|\sum_{i=1}^{m}(w_i(\bar{z}) - w_i(\hat{\bar{z}}))(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u})\| \leq \mu_2\|\boldsymbol{e}_2\|$, since, for this subsystem the observer has already been designed, and the scheduling vector depended on the states to be estimated. When a new subsystem is introduced, both z and Δ change. In order to keep the symmetry and obtain a condition similar to that of centralized observer design, in this paper we require that Δ is expressed as

$$\sum_{i=1}^{m}(w_i(z) - w_i(\hat{z}))\begin{pmatrix} A_{1i}\boldsymbol{x}_1 + B_{1i}\boldsymbol{u} + A_{12i}\boldsymbol{x}_1 \\ A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1 \end{pmatrix} = \begin{pmatrix} F_1 & F_{12} \\ F_{21} & F_2 \end{pmatrix}\boldsymbol{e} \tag{35}$$

and that the uncertainties corresponding to different parts of the error are bounded:

$$\begin{aligned} &\|F_{12}\| \le \mu_{12} \\ &\|F_1\| \le \mu_1 \\ &\|F_{21}\| \le \mu_{21} \\ &\|F_2\| \le \mu_2 \end{aligned} \tag{36}$$

Considering a distributed observer design for such a system, the following stability conditions can be formulated:

Corollary 5. *The error system* (33), *with the properties* (35) *and* (36), *is asymptotically stable, if there exist* $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $L_{1i}, L_{2i}, i = 1, 2, \dots, m$ *so that*

$$\mathcal{H}(P_2(G_{2i} + F_2)) < -2Q_2, \quad i = 1, 2, \dots, m$$

$$\mathcal{H}(P_1 G_{1i}) < -2Q_1, \quad i = 1, 2, \dots, m$$

$$\lambda_{\min}(\mathcal{H}(Q_1 + P_1 F_1)) > \max_i \|P_1(G_{12i} + F_{12})\|$$

$$\frac{\lambda_{\min}(\mathcal{H}(P_1 G_{1i} + Q_1))}{\max_i \|P_1(G_{12i} + F_{12})\|} > \frac{\max_i \|P_2(G_{21i} + F_{21})\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2(G_{2i} + F_2) + Q_2))}$$

where $G_{1i} = A_{1i} - L_{1i}C_{11}$, $G_{2i} = A_{2i} - L_{2i}C_{21}$, $G_{12i} = A_{12i} - L_{1i}C_{12}$, $G_{21i} = A_{21i} - L_{2i}C_{21}$.

Proof. Consider first the following part of the system (33):

$$\dot{\boldsymbol{e}}_c = \sum_{i=1}^{m} w_i(\hat{z})\begin{pmatrix} (A_{1i} - L_{1i}C_{11})\boldsymbol{e}_{1c} \\ A_{2i}\boldsymbol{e}_{2c} + A_{21i}\boldsymbol{e}_{1c} - L_{2i}C_2\boldsymbol{e}_c \end{pmatrix} + \sum_{i=1}^{m}(w_i(z) - w_i(\hat{z}))\begin{pmatrix} 0 \\ A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1 \end{pmatrix} \tag{37}$$

This is a cascaded system and it is asymptotically stable, if the conditions of Theorem 3 are satisfied. First, we show that system (37) is exponentially stable, if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, $\mu_2 \ge 0$, $\mu_{21} \ge 0$, F_2, F_{21} so that

$$\mathcal{H}(P_1 G_{1i}) < -2Q_1, \quad i = 1, 2, \dots, m$$

$$\sum_{i=1}^{m}(w_i(z) - w_i(\hat{z}))(A_{2i}\boldsymbol{x}_2 + B_{2i}\boldsymbol{u} + A_{21i}\boldsymbol{x}_1) = (F_{21} \quad F_2)\boldsymbol{e}_c$$

$$\|F_{21}\| \le \mu_{21}$$

$$\|F_2\| \le \mu_2$$

$$\mathcal{H}(P_2(G_{2i} + F_2)) < -2Q_2, \quad i = 1, 2, \dots, m$$

with $G_{1i} = A_{1i} - L_{1i}C_{11}$ and $G_{2i} = A_{2i} - L_{2i}C_{21}$.

The condition $\mathcal{H}(P_2(G_{2i} + F_2)) < -2Q_2$ ensures that the already existing error system is exponentially stable. Moreover, there exists $\alpha \in \mathcal{R}^+$ so that $V_c = \boldsymbol{e}_c^T \operatorname{diag}(\alpha P_1, P_2)\boldsymbol{e}_c$ is a Lyapunov function for (37) and $\dot{V}_c < -2\boldsymbol{e}_c^T Q\boldsymbol{e}_c$, with $Q = \operatorname{diag}(\alpha Q_1, Q_2)$ and $G_{21i} = A_{21i} - L_{2i}C_{21}$. To prove this, consider the Lyapunov function

$$V_c = \boldsymbol{e}_c^T \begin{pmatrix} \alpha P_1 & 0 \\ 0 & P_2 \end{pmatrix} \boldsymbol{e}_c$$

The derivative can be computed as

$$\dot{V}_c = \sum_{i=1}^{m} w_i(\widehat{z}) e_c^T \mathcal{H} \begin{pmatrix} \alpha P_1 G_{1i} & 0 \\ P_2(G_{21i} + F_{21}) & P_2(G_{2i} + F_2) \end{pmatrix} e_c$$

For $\dot{V}_c < -2e_c^T Q e_c$, it is required that

$$\begin{pmatrix} \alpha\mathcal{H}(P_1 G_{1i}) & (G_{21i} + F_{21})^T P_2 \\ P_2(G_{21i} + F_{21}) & \mathcal{H}(P_2(G_{2i} + F_2)) \end{pmatrix} < -2 \begin{pmatrix} \alpha Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

which amounts to

$$\begin{pmatrix} \alpha\mathcal{H}(P_1 G_{1i} + Q_1) & (G_{21i} + F_{21})^T P_2 \\ P_2(G_{21i} + F_{21}) & \mathcal{H}(P_2(G_{2i} + F_2) + Q_2) \end{pmatrix} < 0$$

Using the Schur complement, we obtain

$$\alpha\mathcal{H}(P_1 G_{1i} + Q_1) - (G_{21i} + F_{21})^T P_2(\mathcal{H}(P_2(G_{2i} + F_2) + Q_2))^{-1} P_2(G_{21i} + F_{21}) < 0$$

which is satisfied by any α chosen such that

$$\alpha > \frac{1}{\lambda_{\min}(\mathcal{H}(P_1 G_{1i} + Q_1))} \cdot \frac{\max_i \|P_2(G_{21i} + F_{21})\|^2}{\lambda_{\min}(\mathcal{H}(P_2(G_{2i} + F_2) + Q_2))} \tag{38}$$

Now, consider the full error system (33), together with the assumptions

$$\sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z}))(A_{1i}x_1 + B_{1i}u + A_{12i}x_1) = (F_1 \quad F_{12})e$$
$$\|F_{12}\| \le \mu_{12}$$
$$\|F_1\| \le \mu_1 \tag{39}$$

Note that these assumptions, together with

$$\sum_{i=1}^{m}(w_i(z) - w_i(\widehat{z}))(A_{2i}x_2 + B_{2i}u + A_{21i}x_1) = (F_{21} \quad F_2)e_c$$
$$\|F_{21}\| \le \mu_{21}$$
$$\|F_2\| \le \mu_2 \tag{40}$$

are effectively equivalent to those that would be used in the centralized design (see Theorem 2).

By using the above constructed $V = V_c$ as a candidate Lyapunov function for (33), we obtain

$$\dot{V} = \sum_{i=1}^{m} w_i(\widehat{z}) e^T \left(\begin{pmatrix} \alpha\mathcal{H}(P_1 G_{1i}) & G_{21i}^T P_2 \\ P_2 G_{21i} & \mathcal{H}(P_2 G_{2i}) \end{pmatrix} + \begin{pmatrix} 0 & \alpha P_1 G_{12i} \\ \alpha G_{12i}^T P_1 & 0 \end{pmatrix} \right) e + e^T \begin{pmatrix} \alpha\mathcal{H}(P_1 F_1) & \alpha P_1 F_{12} \\ \alpha(P_1 F_{12})^T & 0 \end{pmatrix} e$$
$$< -e^T \mathcal{H} \begin{pmatrix} \alpha(Q_1 + P_1 F_1) & 0 \\ 0 & Q_2 \end{pmatrix} e + 2e^T \left[\alpha \max_i \|P_1(G_{12i} + F_{12})\| \right] e$$
$$< -e^T \begin{pmatrix} \alpha\mathcal{H}(Q_1 + P_1 F_1 - \alpha \max_i \|P_1(G_{12i} + F_{12})\|) & 0 \\ 0 & \mathcal{H}(Q_2 - \alpha \max_i \|P_1(G_{12i} + F_{12})\|) \end{pmatrix} e$$

which leads to the conditions

$$\lambda_{\min}(\mathcal{H}(Q_1 + P_1 F_1)) > \max_i \|P_1(G_{12i} + F_{12})\| \tag{41}$$

$$\lambda_{\min}(Q_2) > \alpha \max_i \|P_1(G_{12i} + F_{12})\| \tag{42}$$

Combining (38) and (42), we find that α exists and $V = V_c$ is a Lyapunov function if

$$\frac{\lambda_{\min}(Q_2)}{\max_i \|P_1(G_{12i}+F_{12})\|} > \frac{\max_i \|P_2(G_{21i}+F_{21})\|^2}{\lambda_{\min}(\mathcal{H}(P_1G_{1i}+Q_1))\lambda_{\min}(\mathcal{H}(P_2(G_{2i}+F_2)+Q_2))}$$

or

$$\frac{\lambda_{\min}(\mathcal{H}(P_1G_{1i}+Q_1))}{\max_i \|P_1(G_{12i}+F_{12})\|} > \frac{\max_i \|P_2(G_{21i}+F_{21})\|^2}{\lambda_{\min}(Q_2)\lambda_{\min}(\mathcal{H}(P_2(G_{2i}+F_2)+Q_2))} \qquad \square$$

Note that for this case (i.e., when the scheduling vector depends on states to be estimated), a cascaded error system can only be obtained in special cases. As in Section 4.2, the conditions of Corollary 5 can be implemented in a two-step algorithm, similar to Algorithm 2. In order to facilitate the design of observer for the next subsystem, the appropriately modified conditions of Corollary 1 can be used. If a bound on the interconnection terms is known in advance, a decoupled design is also possible, similar to that given in Theorem 6.

5. Examples

In this section we illustrate on two examples how our approach can be applied for stability analysis and observer design, respectively. The first example presents sequential stability analysis for a nearly cascaded system, while the second one illustrates sequential observer design.

5.1. Stability analysis

First we present a numerical example to illustrate sequential stability analysis. This example involves a nearly cascaded system, i.e., a system where one interconnection terms is strong, and the other one is weak. Note that for such systems, sequential stability analysis is in general better suited than the parallel approaches that can be found in the literature. The explanation is that for a nearly cascaded system, the subsystem influenced by the strong interconnection is not able to dominate the interconnection term. However, this is not the case in sequential analysis, where one subsystem has to handle the product of the interconnection terms.

Consider a distributed system consisting of two subsystems, as follows:

Subsystem 1:

If z is small, then

$$\dot{\boldsymbol{x}}_1 = A_{11}\boldsymbol{x}_1 = \begin{pmatrix} -1 & 1 \\ 1 & -5 \end{pmatrix}\boldsymbol{x}_1$$

If z is large, then

$$\dot{\boldsymbol{x}}_1 = A_{12}\boldsymbol{x}_1 = \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}\boldsymbol{x}_1$$

Subsystem 2:

$$\dot{x}_2 = A_2x_2 = -10x_2$$

The interconnection terms are given as $f_{21} = [0.1 \quad 0]^T x_2$ and $f_{12} = [5 \quad 0]\boldsymbol{x}_1$.

This system is "nearly cascaded", and stable, which is provable with a common Lyapunov function of the form $V = [\boldsymbol{x}_1^T \quad x_2]P[\boldsymbol{x}_1^T \quad x_2]^T$. Note that f_{12} is a strong connection between the subsystems, while f_{21} is a weak connection. Using the stability analysis of [29], we obtain the conditions following:

$$\begin{aligned} &\mathcal{H}(P_1A_{11}) < -Q_1 \\ &\mathcal{H}(P_1A_{12}) < -Q_1 \\ &\lambda_{\min}(Q_1) \geq 2 + 25\|P_1^2\| \end{aligned} \qquad (43)$$

and

$$\begin{aligned}&\mathcal{H}(P_2A_2) < -Q_2\\&\lambda_{\min}(Q_2) \geq 2 + 0.01\|P_2^2\|\end{aligned}$$

respectively. According to Matlab's *feasp*, condition (43) is unfeasible, and therefore stability cannot be proven with the method of [29].

Using the conditions of Theorem 4, derived in this paper, we have

$$\begin{aligned}&\mathcal{H}(P_1A_{11}) < -2Q_1\\&\mathcal{H}(P_1A_{12}) < -2Q_1\\&\gamma = 135.98\\&\mathcal{H}(P_2A_2) < -2Q_2\\&Q_2 \geq 0.1P_2\\&-2(P_2A_2 + Q_2) > 0.1\gamma P_2\end{aligned}$$

which is feasible. Hence, in this case, the proposed sequential stability analysis is better suited than approaches in the literature (see also Remark 3.1 in [29], stating that in case of distributed stability analysis, the interconnection terms for all subsystems should be "small enough").

5.2. Observer design

In this example, we illustrate the sequential observer design for a real-world system.

Consider a distributed system where each individual subsystem is a cascaded tanks system as shown in Fig. 4.

This system is described as follows: water is pumped into the upper tank **1** that has a cross-sectional area A_1. The level of the water in this tank is denoted by h_1. From this tank, the water flows out through a pipe with cross-sectional area s_1 into the lower tank **2** that has a cross-sectional area A_2. The level of the water in this tank is h_2. From the lower tank, the water flows out through a pipe with cross-sectional area s_2 into a reservoir.

The system has one control input, u, which is the voltage applied to the motor that pumps water into the upper tank with a flow rate F_{in}. The measured output is the water level h_2 in the lower tank.

The dynamics of this system are described by

$$\begin{aligned}\tau_1\dot{F}_{\text{in}} &= -F_{\text{in}} + Q_s \cdot u\\ \dot{h}_1 &= \frac{F_{\text{in}}}{A_1} - \frac{s_1\sqrt{2gh_1}}{A_1}\\ \dot{h}_2 &= \frac{s_1\sqrt{2gh_1}}{A_2} - \frac{s_2\sqrt{2gh_2}}{A_2}\end{aligned} \tag{44}$$

where Q_s is the input-to-flow gain, τ is the motor time constant, and g is the acceleration due to gravity.

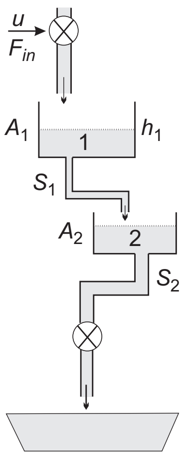

Fig. 4. A cascaded tanks system.

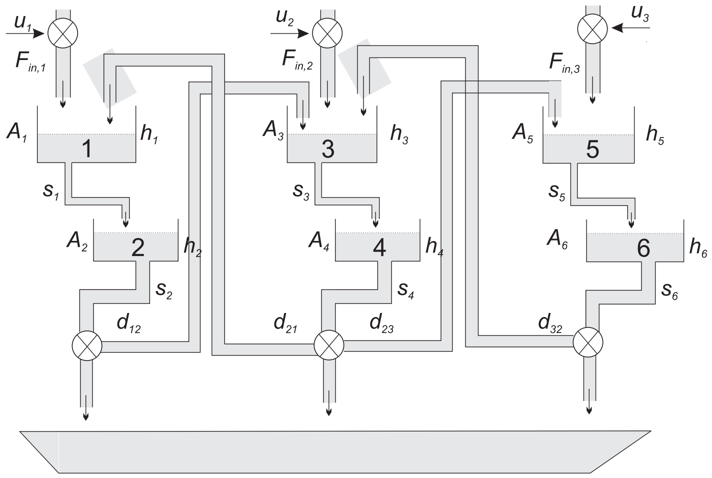

Fig. 5. Coupled cascaded tanks system.

In a distributed system, several of these individual cascaded tanks systems are interconnected. For instance, a system with three subsystems is shown in Fig. 5. The interconnection between the subsystems consists in redistributing part of the water that would flow to the reservoir to the neighboring tanks, indicated in Fig. 5 by the links d_{12}, d_{21}, d_{23}, and d_{32}.

In case of the system presented in Fig. 5, water is pumped into the upper tanks 1, 3, and 5. From these tanks, the water flows to the lower tanks 2, 4, and 6. From the lower tanks, part of the water flows into a reservoir, and part is redistributed to the neighboring tanks. Each cascaded tank system has one control input u_i, which is the voltage applied to the motor of the corresponding pump, and one measured output: the water level in the lower tank. The measured outputs for the whole system are therefore h_2, h_4, and h_6. The flow rates $F_{\text{in},i}$, provided by the pumps, and the water levels in the upper tanks have to be estimated, and therefore, an observer has to be designed. The dynamics of the distributed system after all the subsystems have been added is given by

$$\begin{aligned}
\tau_1 \dot{F}_{\text{in},1} &= -F_{\text{in},1} + Q_{s,1} \cdot u_1 \\
\dot{h}_1 &= \frac{F_{\text{in},1}}{A_1} - \frac{s_1\sqrt{2gh_1}}{A_1} + d_{21}\frac{s_4\sqrt{2gh_4}}{A_1} \\
\dot{h}_2 &= \frac{s_1\sqrt{2gh_1}}{A_2} - \frac{s_2\sqrt{2gh_2}}{A_2} \\
\tau_2 \dot{F}_{\text{in},2} &= -F_{\text{in},2} + Q_{s,2} \cdot u_2 \\
\dot{h}_3 &= \frac{F_{\text{in},2}}{A_3} - \frac{s_3\sqrt{2gh_3}}{A_3} + d_{12}\frac{s_2\sqrt{2gh_2}}{A_3} + d_{32}\frac{s_6\sqrt{2gh_6}}{A_3} \\
\dot{h}_4 &= \frac{s_3\sqrt{2gh_3}}{A_4} - \frac{s_4\sqrt{2gh_4}}{A_4} \\
\tau_3 \dot{F}_{\text{in},3} &= -F_{\text{in},3} + Q_{s,3} \cdot u_3 \\
\dot{h}_5 &= \frac{F_{\text{in},3}}{A_5} - \frac{s_5\sqrt{2gh_5}}{A_5} + d_{23}\frac{s_4\sqrt{2gh_4}}{A_5} \\
\dot{h}_6 &= \frac{s_5\sqrt{2gh_5}}{A_6} - \frac{s_6\sqrt{2gh_6}}{A_6}
\end{aligned} \tag{45}$$

where $d_{21}(s_4\sqrt{2gh_4}/A_1)$, $d_{12}(s_2\sqrt{2gh_2}/A_3)$, $d_{32}(s_6\sqrt{2gh_6}/A_3)$, and $d_{23}(s_4\sqrt{2gh_4}/A_5)$ represent the interconnection terms between the subsystems, i.e., the amount of water redistributed among the tanks.

Table 1
Parameter values.

Parameter	Symbol	Value	Units
Acceleration due to gravity	g	9.81	$\mathrm{m/s^2}$
Cross-sectional area of tanks	$A_1, A_2, A_3, A_4, A_5, A_6$	10, 11, 11, 9, 12, 10	$\mathrm{m^2}$
Outlet area of tanks	$s_1, s_2, s_3, s_4, s_5, s_6$	0.2, 0.1, 0.2, 0.125, 0.25, 0.135	$\mathrm{m^2}$
Input to flow gains	Q_{s1}, Q_{s2}, Q_{s3}	33.3	$\mathrm{m^3/s/V}$
Motor time constants	τ_1, τ_2, τ_3	3	s
Distribution ratios	$d_{12}, d_{21}, d_{23}, d_{31}$	0.3, 0.1, 0.1, 0.3	–
Minimum water level	$h_{\min}$	0.2	m
Maximum water level	$h_{\max}$	2	m

Our goal is to design observers to estimate the flow rates $F_{\mathrm{in},i}$, $i = 1, 2, 3$, and the water level in the upper tanks h_1, h_3, and h_5. It is assumed that the subsystems are added one after another, and, moreover, further subsystems can be added to this system. Therefore, the design is performed sequentially.

The parameter values that will be used for simulation purposes are presented in Table 1.

It is assumed that the tanks have the same height, $h_{\max} = 2$ m, and the water level in the tanks cannot drop below $h_{\min} = 0.2$ m. Therefore, all levels are bounded, $h_i \in [h_{\min}, h_{\max}]$, $i = 1, 2, 3, 4, 5, 6$.

In order to use the proposed design, we construct the distributed TS fuzzy model sequentially, as each subsystem is being added: at $t = t_0$ only the first system exists, to which at $t = t_1$ the second one is added (with the interconnections), to which at $t = t_2$ the third cascaded system (with interconnections) is added, etc. Therefore, consider the first cascaded tanks system with tanks 1 and 2, with the dynamics given by (44).

An exact fuzzy representation of the model (44) can be obtained using the sector nonlinearity approach [48]. Using this approach, we obtain a four-rule TS fuzzy system, with the local matrices[2]

$$A_{11} = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.19 & 0 \\ 0 & 0.18 & -0.09 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.19 & 0 \\ 0 & 0.18 & -0.03 \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 & -0.06 & 0 \\ 0 & 0.05 & -0.09 \end{pmatrix}, \quad A_{14} = \begin{pmatrix} -0.33 & 0 & 0 \\ 0.10 - 0.06 & 0 & \\ 0 & 0.05 & -0.03 \end{pmatrix}$$

scheduling variables h_1 and h_2, weighting functions $\eta_1^0 = \left(\sqrt{0.2}\big/\sqrt{h_1}\right)\left(\sqrt{2} - \sqrt{h_1}\right)\big/\left(\sqrt{2} - \sqrt{0.2}\right)$, $\eta_1^1 = 1 - \eta_1^0$, $\eta_2^0 = \left(\sqrt{0.2}\big/\sqrt{h_2}\right)\left(\sqrt{2} - \sqrt{h_2}\right)\big/\left(\sqrt{2} - \sqrt{0.2}\right)$, $\eta_2^1 = 1 - \eta_2^0$, and membership functions $w_1 = \eta_1^0\eta_2^0$, $w_2 = \eta_1^0\eta_2^2$, $w_3 = \eta_1^2\eta_2^0$, and $w_4 = \eta_1^2\eta_2^2$. Note that due to the nature of the interconnections, by the addition of a new subsystem, these local matrices will not change. In turn, the nonlinearity in the interconnection term leads to two more weighting functions, and therefore the membership functions do change. However, Assumption 1 is satisfied. To design the observer, we follow the steps of Algorithm 2. It has to be noted is that the membership functions depend on h_1, which is a state that has to be estimated. The observer–model mismatch for this subsystem is bounded by $\mu_1 = 0.099$.

To design the observer, we solve[3] the following LMI problem: *find* $P_1 = P_1^T > 0$, $Q_1 = Q_1^T > 0$, $R = R^T > 0$, M_i, $i = 1, 2, 3, 4$, *so that*

$$\begin{aligned} &\mathcal{H}(P_1 A_{1i} - M_i C) < -4Q_1 - R \\ &\begin{pmatrix} R - \mu_1^2 & P_1 \\ P_1 & I \end{pmatrix} > 0 \end{aligned} \tag{46}$$

[2] All values are rounded to two decimal places.
[3] The LMIs in this section have been solved using the SeDuMi solver within the Yalmip toolbox [49] for Matlab.

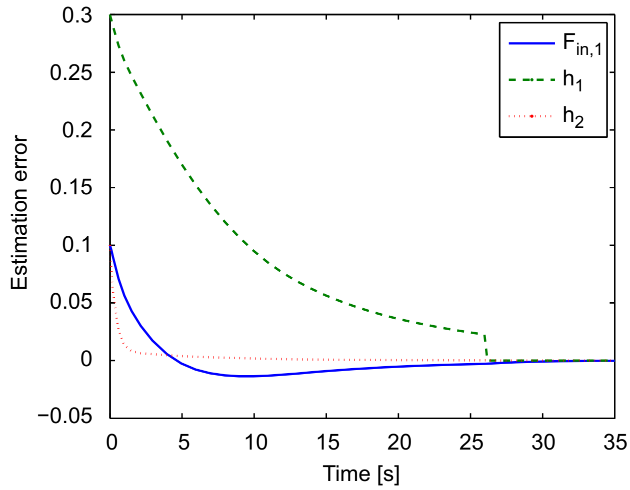

Fig. 6. Estimation error for one subsystem.

and we obtain

$$P_1 = 0.1I, \quad Q_1 = 9.84 \times 10^{-4} I, \quad R = \begin{pmatrix} 0.06 & -0.01 & -0.00 \\ -0.01 & 0.02 & 0.00 \\ -0.00 & 0.00 & 0.27 \end{pmatrix}.$$

The observer gains are computed as $L_{1i} = P_1^{-1} M_i$ and we obtain the values

$$L_{11} = \begin{pmatrix} -0.01 \\ 0.18 \\ 2.67 \end{pmatrix}, \quad L_{12} = \begin{pmatrix} 2.68 \\ 2.86 \\ 2.70 \end{pmatrix}, \quad L_{13} = \begin{pmatrix} 2.68 \\ 2.80 \\ 2.67 \end{pmatrix}, \quad L_{14} = \begin{pmatrix} 2.68 \\ 2.80 \\ 2.70 \end{pmatrix}$$

Moreover, we have $\lambda_{\min}(Q_1) = 9.84 \times 10^{-4}$, $\lambda_{\min}(\mathcal{H}(P_1 A_{1i} - M_i C + Q) + R) = 1.98 \times 10^{-3}$, $\|P_1\| = 0.1$, and $\bar{\gamma_1} = 5.1704 \times 10^3$ (see Algorithm 2). As illustrated in Fig. 6, this observer correctly estimates the states of the first subsystem. The trajectory presented in Fig. 6 has been obtained for a randomly generated input vector, u drawn from the uniform distribution $\mathcal{U}[0, 0.1]$, the true initial states being $\boldsymbol{x}_0 = (0.1\ 0.5\ 0.3)^T$ and the estimated initial states being $\hat{\boldsymbol{x}}_0 = (0\ 0.2\ 0.2)^T$.

At the time $t = t_1$ the second subsystem (the cascaded tank system with tanks 3 and 4) is added, together with the interconnection terms to and from the first subsystem (d_{12} and d_{21} in Fig. 5). The upper bound on these interconnection terms are $\upsilon_{21} = \|d_{21}(s_4\sqrt{2gh_4}/A_1)\| \leq 0.0078$ and $\upsilon_{12} = \|d_{12}(s_2\sqrt{2gh_2}/A_3)\| \leq 0.171$. With these bounds, we have $\gamma_1 = 1.50$. An exact TS representation of this second subsystem is obtained using the sector nonlinearity approach, similar to the TS model of the first subsystem. Since one of the scheduling variables is h_3, which is a state that has to be estimated, again an observer–model mismatch that is bounded by $\mu_2 = 0.11$, appears. Since the interconnection terms depend on measured variables, we have that $\mu_{12} = 0$ and $\mu_{21} = 0$. Note that although there will be eight rules, there are only four different matrices that concern the individual dynamics of the second subsystem (the other four are due to the nonlinearity in the interconnection term). Moreover, as already stated, the introduction of the interconnection term does not change the local matrices of the existing subsystem.

To design the observer for this second subsystem, we solve the problem *find* $P_2 = P_2^T > 0$, $Q_2 = Q_2^T > 0$, $R_2 = R_2^T > 0$, M_i, $i = 1, 2, 3, 4$ *so that*

$$\mathcal{H}(P_2 A_{2i} - M_i C) < -4Q_2 - R$$

$$\begin{pmatrix} R - \mu_2^2 & P_2 \\ P_2 & I \end{pmatrix} > 0$$

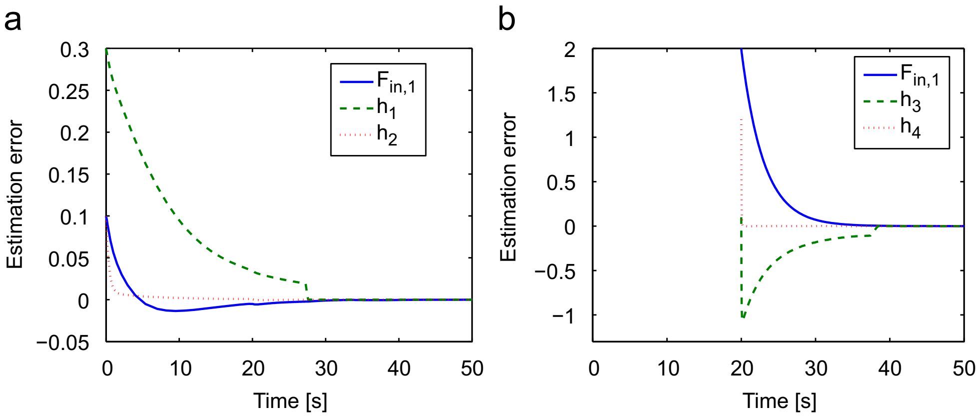

Fig. 7. Simulation results for the addition of the second subsystem. (a) Estimation error for subsystem 1. (b) Estimation error for subsystem 2.

$$\begin{pmatrix} (1-\beta)Q_2 & \upsilon_{12}P_2 \\ \upsilon_{12}P_2 & (1-\beta)Q_2 \end{pmatrix} > 0$$

$$\begin{pmatrix} (1-\beta)Q_2 & \gamma_1\upsilon_{12}P_2 \\ \gamma_1\upsilon_{12}P_2 & (1-\beta)Q_2 \end{pmatrix} > 0 \tag{47}$$

The goal is to be able to solve the inequalities above for the largest $\beta \in [0, 1]$ possible. This can be done either by using a BMI solver (e.g., Penbmi [50]) or by a line search on β. For a fixed β, (47) becomes an LMI problem. For this example, we solved it for $\beta = 0.7$, and obtained

$$P_2 = \begin{pmatrix} 77.50 & 16.72 & -16.84 \\ 16.72 & 46.27 & -45.87 \\ -16.84 & -45.87 & 47.59 \end{pmatrix}, \quad Q_2 = 5.14I,$$

and the observer gains computed as $L_{2i} = P_2^{-1}M_i$. An error trajectory is illustrated in Figs. 7(a) and (b). The second subsystem has been added at $t_1 = 20$s. The true initial states for the second subsystem were $\boldsymbol{x}_0 = (2\ 0.3\ 1.4)^T$ and the estimated initial states $\hat{\boldsymbol{x}}_0 = (0\ 0.2\ 0.2)^T$. The input to the system has been randomly generated from $\mathcal{U}[0, 0.1]$.

In the simulation, instead of the measured values, the estimated values of h_2 and h_4 have been used in the second and first subsystem, respectively. Consequently, due to the incorrect estimate of the initial state, the addition of the new subsystem influences the estimation error of the existing subsystem. This can be seen in Fig. 8.

With the two observers designed, we can compute α to determine the Lyapunov function for the combined error dynamics, and we obtain $\alpha = 1.44 \times 10^{-4}$. Consequently, the Lyapunov matrix for the first two subsystems is $P_{12} = \left(\begin{smallmatrix} \alpha P_2 & 0 \\ 0 & P_1 \end{smallmatrix}\right)$, with $\|P_{12}\| = \max\{\alpha\|P_2\|, \|P_1\|\} = 0.1$, and $Q_{12} = \beta\left(\begin{smallmatrix} \alpha Q_2 & 0 \\ 0 & Q_1 \end{smallmatrix}\right)$, with $\lambda_{\min}(Q_{12}) = \beta\min\{\alpha\lambda_{\min}(Q_2), \lambda_{\min}(Q_1)\} = 5.19 \times 10^{-4}$. Moreover, we also have that $\mathcal{H}(P_{12}A_i^{12} - M_i^{12}C + Q_{12}) + R_{12} < -2Q_{12}$, and we can compute $\bar{\gamma}_2 = 1.8534 \times 10^4$.

At the time $t = t_2$ the third subsystem (the cascaded tank system with tanks 5 and 6) is added, together with the interconnection terms to and from the second subsystem (d_{32} and d_{23} in Fig. 5). The upper bound on these interconnection terms are $\upsilon_{32} = \|d_{32}(s_6\sqrt{2gh_6}/A_3)\| \leq 0.0231$, and $\upsilon_{23} = \|d_{23}(s_4\sqrt{2gh_4}/A_5)\| \leq 0.0065$. With these bounds, we have $\gamma_2 = 0.7891$. An exact TS representation of this subsystem is obtained similar to the previous ones. The observer-model mismatch is bounded by $\mu_3 = 0.13$. To design the observer, we solved the LMI problem[4] *find*

[4] Similar to the design for the second subsystem, this can also be formulated as a BMI problem, with both β and Q_3 decision variables.

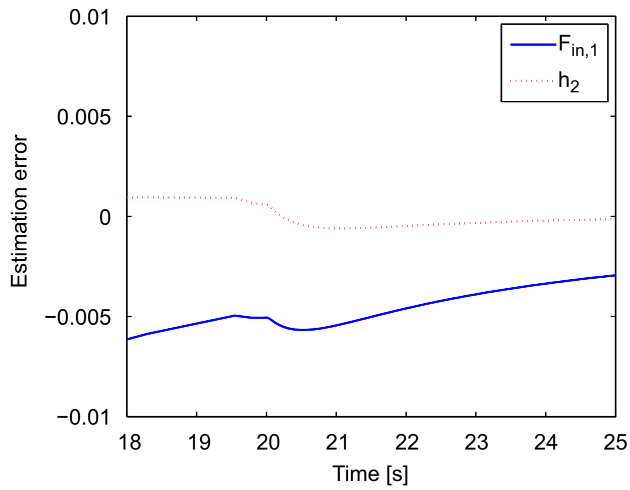

Fig. 8. Influence of the addition of the new subsystem.

$P_3 = P_3^T > 0$, $Q_3 = Q_3^T > 0$, $R = R^T > 0$, M_i, $i = 1, 2, 3, 4$, *so that*

$$\begin{aligned}
&\mathcal{H}(P_3 A_{3i} - M_i C) < -4Q_3 - R\\
&\begin{pmatrix} R - \mu_3^2 & P_3 \\ P_3 & I \end{pmatrix} > 0\\
&\begin{pmatrix} (1-\beta)Q_3 & \upsilon_{23} P_3 \\ \upsilon_{23} P_3 & (1-\beta)Q_3 \end{pmatrix} > 0\\
&\begin{pmatrix} (1-\beta)Q_3 & \gamma_2 \upsilon_{23} P_3 \\ \gamma_2 \upsilon_{23} P_3 & (1-\beta)Q_3 \end{pmatrix} > 0
\end{aligned} \tag{48}$$

for $\beta = 0.3$, and obtained

$$P_2 = \begin{pmatrix} 14.34 & 2.02 & -2.38 \\ 2.02 & 10.29 & -9.70 \\ -2.38 & -9.70 & 11.53 \end{pmatrix}, \quad Q_2 = 1.15I,$$

and the observer gains were computed as $L_{3i} = P_3^{-1} M_i$. An error trajectory of this subsystem is illustrated in Fig. 9. The subsystem has been added at $t_1 = 40$ s. The true initial states were $\boldsymbol{x}_0 = (1\ 0.25\ 0.6)^T$ and the estimated initial states $\hat{\boldsymbol{x}}_0 = (0\ 0.2\ 0.2)^T$. The input to the system has been randomly generated from $\mathcal{U}[0, 0.1]$.

With the third observer designed, we have $\alpha_2 = 0.0022$, $\|P_{13}\| = \max\{\alpha_2 \|P_3\|, \|P_{12}\|\} = 0.1$, and $\lambda_{\min}(Q_{13}) = \beta \min\{\alpha_2 \lambda_{\min}(Q_3), \lambda_{\min}(Q_{12})\} = 1.66 \times 10^{-5}$. Moreover, we also have that $\bar{\gamma_3} = 1.81 \times 10^7$, and we can start designing an observer for yet another new subsystem.

6. Discussion and conclusions

Many physical systems, such as power systems, communication and distribution networks, economic systems, and traffic networks are composed of interconnections of lower-dimensional subsystems. In this paper, the stability of such distributed systems was investigated for the case when the subsystems are represented as TS fuzzy systems. We considered the case when subsystem are added online, one-by-one to an existing system, and propose conditions for

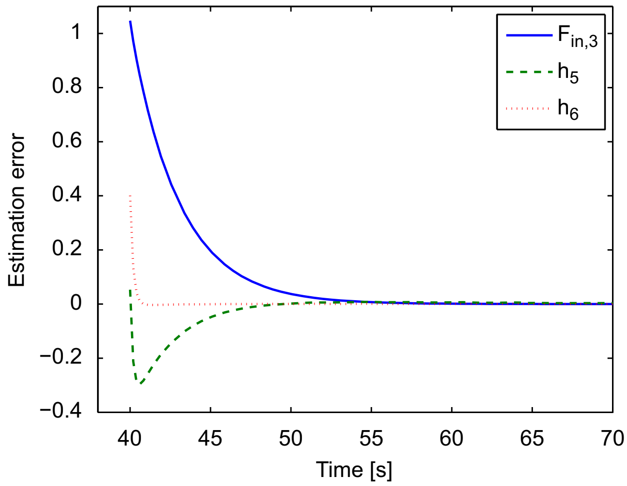

Fig. 9. Estimation error for the third subsystem.

establishing the stability of the whole system when a new subsystem was added. This setting has also been extended to state estimation, by developing a method to sequentially design observers for a distributed system.

The motivation for the sequential analysis and design comes from allowing that subsystems may be added to or removed from a distributed system. Current approaches in the literature assume that the structure of the system is known, that is, the number of subsystems is known a priori. Therefore, subsystems can no longer be added. Moreover, it is often required [5,31,34] that the analysis of the subsystems is performed in parallel, at the same time. To be able to address analysis and design for a distributed system to which subsystems may be added, we have proposed a sequential approach.

Next we discuss some theoretical and practical aspects of the proposed analysis and design methods.

A shortcoming of the method stems from the fact that we consider fairly general nonlinear systems (the only assumption we make is Assumption 1), with any type of interconnection, i.e., we do not exploit a specific structure. Due to this, and due to the fact that Assumption 1 allows for the change of membership functions, we use a membership function independent Lyapunov function, more specifically, a common quadratic Lyapunov function, which renders the result conservative. Another source of conservativeness is the use of a block-diagonal Lyapunov matrix. Using such a matrix ensures the stability of each individual subsystem, and consequently allows the addition or the removal of subsystems (together with the corresponding interconnection terms). If, for instance, subsystems cannot be removed, then a full Lyapunov matrix can be used, and it is no longer a necessary condition that each subsystem is stable. This issue will be addressed in our future research.

From a practical point of view, the main limitation of the proposed methods is given by Assumption 1, i.e., by the assumption that by adding a new subsystem, the individual dynamics of the existing one do not change. For many systems, this assumption is not satisfied. This assumption will be relaxed in our future research, where we will investigate the case when due to the addition of a new subsystem, the dynamics of the individual subsystems change.

A second practical shortcoming is that with each newly added subsystem, the conditions become more and more conservative, in particular, when applying Corollary 1 (or the corresponding design method). We will address this issue in our future research by considering other types of Lyapunov functions, e.g., membership-dependent Lyapunov functions, such as those used in [51,52].

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