Observer design for discrete-time TS fuzzy systems with local nonlinearities

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Abstract— **This paper presents a novel observer design approach for discrete-time nonlinear systems. The nonlinearities are handled on the one hand by a polytopic model and on the other hand with a slope-restricted condition. Using these two tools sufficient LMI conditions are developed for a new observer design technique. The obtained results are tested in simulation and on experimental data on an inverted pendulum on a cart.**

I. Introduction

State estimation is an important problem for real systems, either because it is physically not possible to measure some of the states or because the sensors are too expensive. For this reason, starting from [7] and [10], much research has been done to design state estimators. The model of a dynamic system usually contains a set of nonlinear terms, which needs to be taken into consideration for observer design. Linear approximations can be used, however they only provide local conclusions [8]. Nonlinear observers may be able to satisfy global performances.

An important nonlinear observer design approach was presented in [1] for continuous-time state estimation, and was extended in several papers, see e.g. [2], [3], [15]. This approach assumes that the unmeasured-state nonlinear terms satisfies a slope-restricted condition. A drawback of this condition is that the rest of the error dynamics are considered linear. This restriction is removed in our approach by also considering polytopic models.

Although many results were developed for continuous-time nonlinear systems with slope-restricted nonlinearities, the discrete-time case is not so well explored. In [6] a detailed approach is presented for discrete-time systems with the circle-criterion approach. Other important results on observers for discrete-time Lipschitz nonlinear systems are presented in [16].

In parallel with these results many other observer design methods were developed for nonlinear systems, for example approaches relying on polytopic representations such as Tagaki-Sugeno (TS) fuzzy systems. The fuzzy representation defines the model as a convex combination of local linear models. Conditions in the TS framework are usually formulated as Linear Matrix Inequalities (LMIs), as these are easy to solve with existing convex optimization methods [9]. Among many advantages, like being able to exactly represent a nonlinear model in a convex set of the state-space [13], it has some disadvantages as well. The number of fuzzy-rules/local-models increases exponentially with the number of nonlinearities, and the design problem may become computationally intractable. Another disadvantage concerns the case when the nonlinearities depend on unmeasured-state variables. In this case, classically, the observer-model mismatch has to satisfy a Lipschitz condition, which makes the design conditions more conservative.

In this paper we combine the slope-restricted condition with the TS fuzzy approach to gain the advantages from both, and we apply it for discrete-time systems. With the TS fuzzy representation we extend the slope-restricted approach for polytopic models, and in parallel, we can handle the unmeasured-state nonlinearities. A similar idea was described in [12] and [11] for the continuous-time case. The discrete-time dynamic observer-based feedback control design described in [4] considers a similar observer problem, but there the control is the main target, and the estimation is not so well explored. This paper focuses on the observer design and extends the existing results.

In the sequel, following some notations, the TS fuzzy system with local nonlinearities and the estimation problem are introduced in Section II. Section III presents the main theoretical results. To highlight the main novelty of the paper, a comparison is provided with observer design approach from [4] in Section IV. This is followed by a case study first in simulation in Section V-A and in an experimental setup on an inverted pendulum on a cart in Section V-B. Finally the conclusions and the future directions are presented in Section VI.

Notations. Let $F = F^T \in R^{n\times n}$ be a real symmetric matrix, $F > 0$ and $F < 0$ mean that F is positive definite and negative definite, respectively. I denotes the identity matrix and 0 the zero matrix of appropriate dimensions. The symbol $*$ in a matrix indicates a transposed quantity in the symmetric position, for instance $\begin{pmatrix} P & * \\ A & P \end{pmatrix} = \begin{pmatrix} P & A^T \\ A & P \end{pmatrix}$, $A + * = A + A^T$, and $AP* = APA^T$. The notation $\mathrm{diag}(f_1, ..., f_n)$, where $f_1, ..., f_n \in \mathbb{R}$, stands for the diagonal matrix, whose diagonal components are $f_1, ..., f_n$.

II. Preliminaries and problem statement

The classic TS fuzzy discrete-time model is a convex combination of linear models, having the form:

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$$x(k+1) = \sum_{i=1}^{s} h_i(z(k))(A_i x(k) + B_i u(k))$$
$$y(k) = \sum_{i=1}^{s} h_i(z(k)) C_i x(k), \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $y(k) \in \mathbb{R}^{n_y}$ is the measured output vector, s is the number of rules, $z(k) \in \mathbb{R}^{n_z}$ is the premise vector, and h_i, $i = 1, ..., s$ are nonlinear functions with the property

$$h_i \in [0,\, 1],\, i = 1, ..., s, \quad \sum_{i=1}^{s} h_i(z) = 1. \tag{2}$$

These nonlinear functions are called the membership functions. Matrices A_i, B_i, and C_i represent the $i-th$ local model. Throughout this paper, the following shorthand notation is used to represent convex sums of matrix expressions:

$$F_z = \sum_{i=1}^{s} h_i(z(k)) F_i. \tag{3}$$

Based on this notation, (1) can be rewritten as

$$\begin{aligned} x(k+1) =& A_z x(k) + B_z u(k) \\ y(k) =& C_z x(k). \end{aligned} \tag{4}$$

In order to develop our results we will use the following property and lemmas.

Property 1: (Schur complement). Let $\mathcal{M} = \mathcal{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, with M_{11} and M_{22} square matrices of appropriate dimensions. Then:

$$\begin{aligned} \mathcal{M} < 0 \Leftrightarrow & \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \\ \Leftrightarrow & \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases} \end{aligned} \tag{5}$$

Estimation and control problems are often defined as a double sum negativity problem having the form

$$F_{zz} = \sum_{i=1}^{s} h_i(z) h_j(z) F_{ij} < 0, \tag{6}$$

with symmetric matrices F_{ij}, and nonlinear functions h_i, where $i = 1, ..., s$, satisfying the convex sum property in (2).

Lemma 1 ([14]): Equation (6) is satisfied if the following conditions hold

$$\begin{aligned} F_{ii} <& 0 \\ \frac{2}{s-1} F_{ii} + F_{ij} + F_{ji} <& 0 \quad \forall i, j = 1, ..., s,\, i \neq j. \end{aligned} \tag{7}$$

In order to develop our results we consider the following model structure:

$$\begin{aligned} x(k+1) =& A_z x(k) + G_z \psi(Hx(k)) + f(y(k), u(k)) + S_z d(k) \\ y(k) =& C_z x(k), \end{aligned} \tag{8}$$

where $x(k)$, $u(k)$, $y(k)$ have the same meaning as in (1) and A_z, G_z, S_z and C_z are convex combination of matrices as in (3). With d we denote the disturbance including the unmodelled dynamics, and f is a possibly nonlinear vector function that only depends on measured signals. We assume that the scheduling vector z only depends on measured variables. The nonlinearities that contain unmeasured states are collected in the vector function $\psi(Hx(k))$.

The quantity $\psi(Hx(k)) \in \mathbb{R}^r$ is an r-dimensional vector where $H \in \mathbb{R}^{r \times n_x}$ and each entry is a function of a linear combination of the states, i.e.

$$\psi_i = \psi_i(\sum_{j=1}^{n_x} H_{ij} x_j(k)), \quad i = 1, ..., r.$$

To develop our results, the elements in vector $\psi(Hx(k))$ must fulfill the following assumption.

Assumption 1: For any $i \in \{1, ..., r\}$ there exist constants $0 < b_i < \infty$, so that

$$0 \leq \frac{\psi_i(v) - \psi_i(w)}{v - w} \leq b_i, \quad \forall v, w \in \mathbb{R}, v \neq w. \tag{9}$$

A similar assumption is used in [1], [2], [3], [5].

In view of (9), there exist $\delta_i(k) \in [0,\, b_i]$, so that for any $v,\, w \in \mathbb{R}$

$$\psi_i(v) - \psi_i(w) = \delta_i(k)(v - w). \tag{10}$$

Let $\delta(k) = \mathrm{diag}(\delta_1(k), ..., \delta_r(k))$. To develop our results we consider the following observer structure:

$$\begin{aligned} \hat{x}(k+1) =& A_z \hat{x}(k) + f(y(k), u(k)) + L_z(y(k) - \hat{y}(k)) \\ &+ G_z \psi\left(H\hat{x}(k) + L_\psi(y(k) - \hat{y}(k))\right) \\ \hat{y}(k) =& C_z \hat{x}(k), \end{aligned} \tag{11}$$

where $\hat{x}(k)$ denotes the estimate of $x(k)$, L_z contains the observer gains, and L_ψ is an injection term to obtain a less conservative design. Let us consider the error dynamics, $e(k) = x(k) - \hat{x}(k)$, from where we obtain

$$\begin{aligned} e(k+1) =& (A_z - L_z C_z) e(k) + S_z d(k) \\ &+ G_z \left(\psi(Hx(k)) - \psi(H\hat{x}(k) + L_\psi(y(k) - \hat{y}(k)))\right) \end{aligned} \tag{12}$$

Based on Assumption 1 we can rewrite (12) in the form

$$\begin{aligned} e(k+1) =& (A_z - L_z C_z) e(k) + G_z \delta(k) \eta(k) + S_z d(k) \\ \eta(k) =& (H + L_\psi C) e(k). \end{aligned} \tag{13}$$

III. Main results

In this section we develop sufficient conditions for observer design with the aim of attenuating the effect of the disturbance d. The following Lyapunov function is considered, with $P = P^T > 0$,

$$V(e(k)) = e(k)^T P e(k). \tag{14}$$

Given the estimation error system in (11) we want to find the observer gains L_z and L_ψ so that the following holds:

$$\Delta V \leq -\|e(k)\|^2 + \mu \|d(k)\|^2, \tag{15}$$

where

$$\Delta V := e(k+1)^T P e(k+1) - e(k)^T P e(k). \tag{16}$$

If (15) holds, then based on [4] the estimation error satisfies

$$\|e(k)\| \leq \sqrt{\mu\|d(k)\|^2 + \bar{c}\|e(0)\|^2}, \quad \forall k > 0, \tag{17}$$

where $\bar{c}$ is a constant. The above mentioned inequality is the well known disturbance attenuation condition, where μ is the H_∞ performance index.

In the following theorem we define the main result of this work.

Theorem 1: Consider system (8) and observer (11). If there exist $P = P^T > 0$, $M = M^T = \text{diag}(m_1, ..., m_r) > 0$, N_i, for $i = 1, ..., s$, W_ψ and constant $\mu > 0$, so that

$$\begin{aligned} F_{ii} &< 0 \\ \frac{2}{s-1}F_{ii} + F_{ij} + F_{ji} &< 0 \quad \forall i, j = 1, ..., s,\ i \neq j. \end{aligned} \tag{18}$$

where

$$F_{ij} = \begin{bmatrix} -P+I & * & * & * \\ MH + W_\psi C_i & \nu(M) & * & * \\ 0 & 0 & -\mu I & * \\ PA_i + N_j C_i & PG_i & PS_i & -P \end{bmatrix} \tag{19}$$

and

$$\nu(M) = -2M\text{diag}\left(\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right), \tag{20}$$

then condition (15) is satisfied, and the observer gains can be recovered from $L_i = P^{-1}N_i$ and $L_\psi = M^{-1}W_\psi$.

Proof: We consider the Lyapunov function candidate defined in (14) and the difference in (16). We denote $\zeta(k) = \begin{bmatrix} e(k)^T & (\delta(k)\eta(k))^T & d(k)^T \end{bmatrix}^T$, which leads to

$$\Delta V = \zeta(k)^T \Sigma \zeta(k), \tag{21}$$

where

$$\Sigma = \begin{bmatrix} (A_z + L_z C_z)^T P * -P & * & * \\ G_z^T P(A_z - L_z C_z) & G_z^T P * & * \\ S_z^T P(A_z - L_z C_z) & S_z^T P G_z & S_z^T P * \end{bmatrix} \tag{22}$$

We need to prove that (15) holds. Considering only (22), the design conditions are very conservative. Less conservative design condition can be obtained by adding additional terms:

$$\zeta(k)^T \Sigma\, \zeta(k) + \zeta(k)^T \Gamma\, \zeta(k) \leq -\|e(k)\|^2 + \mu\|d(k)\|^2, \tag{23}$$

where

$$\Gamma = \begin{bmatrix} 0 & * & * \\ MH + ML_\psi C_z & \nu(M) & * \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

Let us now examine $\zeta(k)^T \Gamma \zeta(k)$.

$$\begin{aligned} \zeta(k)^T \Gamma\, \zeta(k) =& (\delta(k)\eta(k))^T \nu(M) * \\ &+ 2e(k)^T (H^T + C_z^T L_\psi^T) M \delta(k)\eta(k) \end{aligned} \tag{25}$$

We know that $\eta(k) = (H_z + L_\psi C_z)e(k)$, which leads to:

$$\zeta(k)^T \Gamma\, \zeta(k) = 2\eta(k)^T \mathcal{D} \eta(k), \tag{26}$$

where $\mathcal{D} = M\delta(k) + 2\delta(k)^T \nu(M)\delta(k)$. Since all the elements in $\mathcal{D}$ are on the main diagonal we can examine the elements:

$$m_i \delta_i(k) \left(1 - \tfrac{1}{b_i}\delta_i(k)\right). \tag{27}$$

Based on (10) $\delta_i(k) \leq b_i$, from where we can conclude that $\mathcal{D} \geq 0$, and this leads to:

$$\zeta(k)^T \Gamma\, \zeta(k) \geq 0. \tag{28}$$

Therefore, if (23) holds then (15) holds as well.

To obtain LMI conditions we consider:

$$\Sigma + \Gamma + \begin{bmatrix} I & * & * \\ 0 & 0 & * \\ 0 & 0 & -\mu I \end{bmatrix} \leq 0, \tag{29}$$

which is equivalent to

$$\begin{bmatrix} \mathcal{A}^T P * + I & * & * \\ G_z^T P \mathcal{A} + \mathcal{B} & G_z^T P * + \nu(M) & * \\ S_z^T P \mathcal{A} & S_z^T P G_z & -\mu I + S_z^T P * \end{bmatrix} \leq 0 \tag{30}$$

where $\mathcal{A} = A_z + L_z C_z$ and $\mathcal{B} = MH + ML_\psi C_z$). In this form (30) is bilinear, to transform it into an LMI, we rewrite in the following form

$$\begin{aligned} &\begin{bmatrix} -P + I & * & * \\ \mathcal{B} & \mu(M) & * \\ 0 & 0 & -\mu I \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{A}^T P \\ G_z^T P \\ S_z^T P \end{bmatrix} P^{-1} \begin{bmatrix} P\mathcal{A} & PG_z & PS_z \end{bmatrix} \leq 0 \end{aligned} \tag{31}$$

We denote $N_z := PL_z$ and $W_\psi := ML_\psi$, and we apply the Schur complement to obtain:

$$\begin{bmatrix} -P+I & * & * & * \\ MH + W_\psi C_z & \nu(M) & * & * \\ 0 & 0 & -\mu I & * \\ PA_z + N_z C_z & PG_z & PS_z & -P \end{bmatrix} \leq 0 \tag{32}$$

Using Lemma 1 sufficient conditions in the form of (18) are obtained. ■

Theorem 1 gives sufficient conditions for (15). The LMI problem can be formulated so that the effect of the disturbance is minimized via the term μ. In the absence of the disturbance the condition provides global asymptotic stability, i.e the error is converging to 0. The following corollary summarizes the conditions for this particular case.

Corollary 1: Consider system (8) with $S_z = 0$, and observer (11). If there exist $P = P^T > 0$, $M = M^T = \text{diag}(m_1, ..., m_r) > 0$, N_i, for $i = 1, ..., s$, W_ψ and constant $\epsilon > 0$ so that

$$\begin{aligned} F_{ii} &< 0 \\ \frac{2}{s-1}F_{ii} + F_{ij} + F_{ji} &< 0 \quad \forall i, j = 1, ..., s,\ i \neq j. \end{aligned} \tag{33}$$

where

$$F_{ij} = \begin{bmatrix} -P + \epsilon I & * & * \\ MH + W_\psi C_i & \nu(M) & * \\ PA_i + N_j C_i & PG_i & -P \end{bmatrix}, \tag{34}$$

and $\nu(M)$ has the same meaning as in Theorem 1, then observer states defined in (11) converge asymptotically to the real system states in (8). The observer gains can be recovered from $L_i = P^{-1}N_i$ and $L_\psi = M^{-1}W_\psi$.

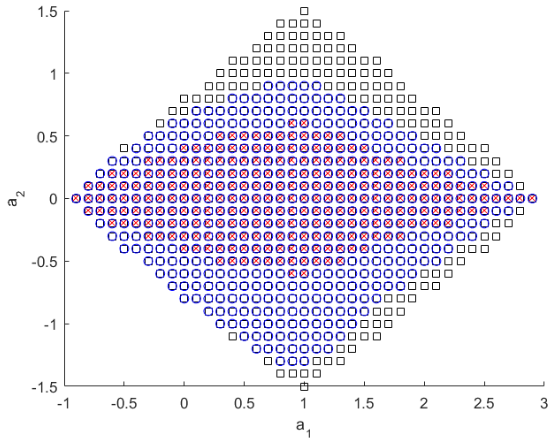

Fig. 1. $b = 1$: 'x'-Theorem 1 from [4], 'o'-Corollary 1 with $W_\psi = 0$, '□'-Corollary 1

Fig. 2. Pendulum on a cart

IV. Comparison with existing results

To highlight the advantages of our approach we compare it with the method for observer design presented in Theorem 1 in [4]. Consider the following example.

Example 1: Model (8) is considered with $f = 0$ and matrices

$$A_1 = A_2 = \begin{bmatrix} 1 & 0.008 \\ 0 & a_1 \end{bmatrix}, \; A_3 = A_4 = \begin{bmatrix} 1 & 0.008 \\ 0 & 0.99 \end{bmatrix}, \; S = 0, \\ G_1 = G_3 = \begin{bmatrix} 0 \\ a_2 \end{bmatrix}, \; G_2 = G_4 = \begin{bmatrix} 0 \\ 0.0027 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{35}$$

Assume that $\psi(Hx(k))$ is scalar and satisfies (9) with $b = 1$ and $H = \begin{bmatrix} 0 & 1 \end{bmatrix}$. For Theorem 1 from [4] we consider $P_{oi} = P$, $S_o = P$, $\theta = b$, and $R = H$. In Theorem 1 from [4] the injection term, L_ψ, is not used, so in the first case we are also omitting that, by considering $W_\psi = 0$ in Corollary 1. We consider the values for $a_1 \in [-0.9, 2.9]$, and for $a_2 \in [-1.5, 1.5]$. The values for which feasible solutions have been obtained can be seen in Fig. 1. Corollary 1 provides a wider range compared to Theorem 1 from [4], and the range can be further extended by considering the injection term L_ψ. A further advantage is that Assumption 1 accepts a wider range of nonlinearities than the Lipschitz condition.

V. Case Study

In this section we consider a pendulum system. First the simulation results are presented, followed by experimental results.

A. Simulation

The continuous-time model of an inverted pendulum on a cart adopted from [11] is considered:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-\gamma x_2 - a(mlx_2)^2 \sin(x_1)\cos(x_1) + mgl\sin(x_1)}{\alpha(x_1)} \\ &\quad + \frac{-aml\cos(x_1)}{\alpha(x_1)}\tilde{u} \\ y &= x_1, \end{aligned} \tag{36}$$

where x_1 is the angle of the pendulum, while x_2 is the angular velocity, $\alpha(x_1) = (J + ml^2) - a(ml\cos(x_1))^2$ and $a = 1/(M+m)$. The rest of the model parameters can be found in Table I.

The model has an unstable equilibrium point at the pointing up position. Since we apply the observer on a real system, we consider the physical limitations as well. For the angle we consider $x_1 \in [-\frac{\pi}{3}, \frac{\pi}{3}]$ and for the angular velocity $x_2 \in [-\sigma, \sigma]$, $\sigma = 24.8$. Model (36) can be transformed to the following form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \rho_1(x_1)x_2 + \rho_2(x_1)\psi(Hx) + \tilde{f}(x_1, u) \\ y &= x_1, \end{aligned} \tag{37}$$

where

$$\begin{aligned} \psi(Hx) &= x_2^2 + 2\sigma x_2 \\ \rho_1(x_1) &= \frac{-\gamma + 2\sigma aml\cos(x_1)\sin(x_1)}{\alpha(x_1)} \\ \rho_2(x_1) &= \frac{-am^2l^2\sin(x_1)\cos(x_1)}{\alpha(x_1)} \\ \tilde{f}(x_1, u) &= \frac{mgl\sin(x_1) - aml\cos(x_1)u}{\alpha(x_1)} \\ H &= \begin{bmatrix} 0 & 1 \end{bmatrix}. \end{aligned} \tag{38}$$

The nonlinearity $\psi(Hx)$ satisfies Assumption 1 with $b = 4\sigma$. We denote $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, and (37) can be written in the matrix form:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & \rho_1(x_1) \end{bmatrix} x + \begin{bmatrix} 0 \\ \rho_2(x_1) \end{bmatrix} \psi(Hx) + \begin{bmatrix} 0 \\ \tilde{f}(y, u) \end{bmatrix}. \tag{39}$$

Based on the real application the sampling time is $T_s = 0.008[s]$, and by using the forward Euler approximation for the derivatives we obtain the following:

$$\begin{aligned} x(k+1) &= A(x_1(k))x(k) + G(x_1(k))\psi(Hx(k)) \\ &\quad + f(y(k), u(k)) \\ y(k) &= Cx(k), \end{aligned} \tag{40}$$

TABLE I
PARAMETER TABLE

Notation	Value	Description
g $[m^s/s]$	9.8	gravitational acceleration
m [kg]	0.2	mass of pendulum
M [kg]	1.61	mass of cart
γ [N/rad/s]	0.4898	friction coefficient
l [m]	0.67	length of pendulum
J $[kg\, m^2]$	0.0232	moment of inertia
K_m [-]	6.5914	PWM gain
σ [rad/s]	24.8	max angular velocity

where

$$\begin{aligned} A(x_1(k)) &= \left(I + T_s \begin{bmatrix} 0 & 1 \\ 0 & \rho_1(x_1(k)) \end{bmatrix}\right), \\ G(x_1(k)) &= T_s \begin{bmatrix} 0 \\ \rho_2(x_1(k)) \end{bmatrix}, \\ f(y(k), u(k)) &= T_s \begin{bmatrix} 0 \\ \tilde{f}(y(k), u(k)) \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \end{aligned} \tag{41}$$

Equation (40) has a form similar to (8), and since $\rho_1(x_1)$ and $\rho_2(x_1)$ depend on x_1, which is measured, based on the sector nonlinearity approach [13] we can transform this into a 4 rule TS fuzzy model, having the following local matrices:

$$\begin{aligned} A_1 &= A_2 = \begin{bmatrix} 1 & 0.008 \\ 0 & 0.8275 \end{bmatrix}, A_3 = A_4 = \begin{bmatrix} 1 & 0.008 \\ 0 & 1.0998 \end{bmatrix}, \\ G_1 &= G_3 = \begin{bmatrix} 0 \\ -0.0027 \end{bmatrix}, G_2 = G_4 = \begin{bmatrix} 0 \\ 0.0027 \end{bmatrix}. \end{aligned} \tag{42}$$

We can notice that in the discrete-time case the local matrices are close to each other. This happens due to the discretization, since the nonlinearities are multiplied with the sampling time. For the simulation we consider the disturbance: $d = [d_1,\, d_2]^T$, and it affects both the angle and the angular velocity in the following way: $S = \begin{bmatrix} 0.1 & 0 \\ 0 & 1 \end{bmatrix}$.

We have applied Theorem 1 and the following observer gains were obtained:

$$\begin{aligned} L_\psi &= -135.03,\ L_1 = \begin{bmatrix} 2.08 \\ 111.96 \end{bmatrix}, L_2 = \begin{bmatrix} 2.08 \\ 111.94 \end{bmatrix}, \\ L_3 &= \begin{bmatrix} 2.08 \\ 148.88 \end{bmatrix}, L_4 = \begin{bmatrix} 2.08 \\ 148.70 \end{bmatrix}. \end{aligned} \tag{43}$$

The LMI problem was defined so that the effect of the disturbance is minimised, and the minimum value is $\mu = 0.046$, so the attenuation will be at least $\sqrt{\mu}$. Note that, similar results can be obtained by considering a linear gain $L_z = L$, but then the value for disturbance attenuation is higher $\mu = 0.057$. In this case the fuzzy observer helps to obtain better performance. A proportional control is applied to stabilize the system in the pointing up position. For the disturbance, d, the standard Gaussian distribution is considered in the following form: $d = 0.1\mathcal{N}(0, 1)$. The initial condition for the model is $x_0 = \begin{bmatrix} 0.75 & -1 \end{bmatrix}^T$, and for the observer $\hat{x}_0 = \begin{bmatrix} 0.75 & 0 \end{bmatrix}^T$. In Fig. 3 we can see the estimated states, $\hat{x}$, while Fig. 4 shows the error signals.

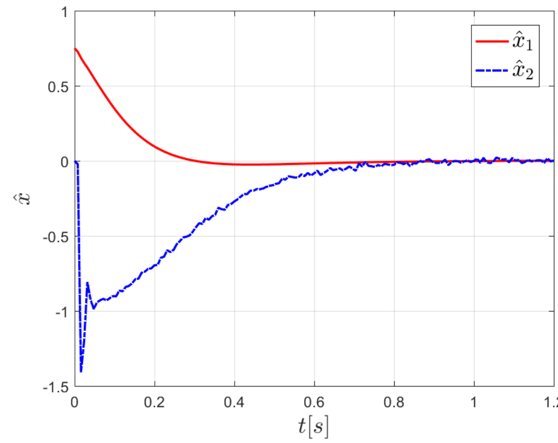

Fig. 3. Estimated state vector

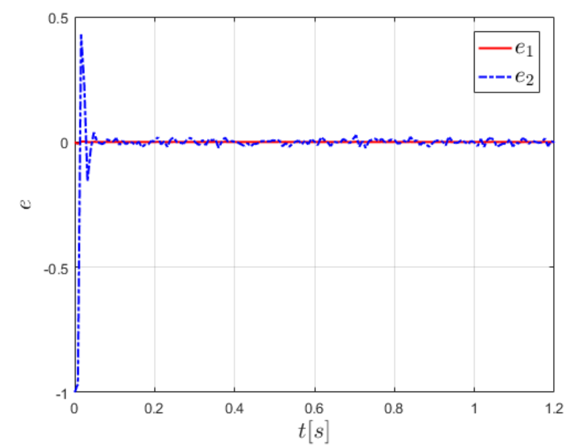

Fig. 4. $x - \hat{x}$

As a comparison, the difference between the estimated angular velocity and the actual angular velocity can be seen in Fig. 5. The observer performs as expected, and even in the presence of the disturbance the estimation is still good.

B. Experimental results

In this part we consider the physical system, of the pendulum on a cart, shown in Fig. 2. The physical limitations for the angle are more strict than in simulation, but for comparison we use the same observer as in Section V-A. The experimental data was collected using a proportional controller, which due to the highly nonlinear dynamics is not able to stabilize the system at the pointing up position, but provides sufficient data to test the observer. Note that the observer-based-control problem will be addressed in future work.

On Fig. 6 we can see the angle, x_1 and the estimated angle $\hat{x}_1$. As it can be seen the estimated angle follows the form of the actual angle with some disturbance on it. The estimated angular velocity can be seen in Fig. 7. It was assumed that the angular velocity is not exceeding the domain $x_2 \in [-\sigma, \sigma]$,

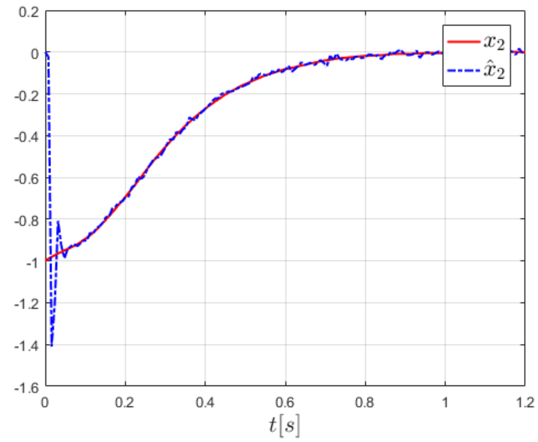

Fig. 5. Actual and estimated angular velocity

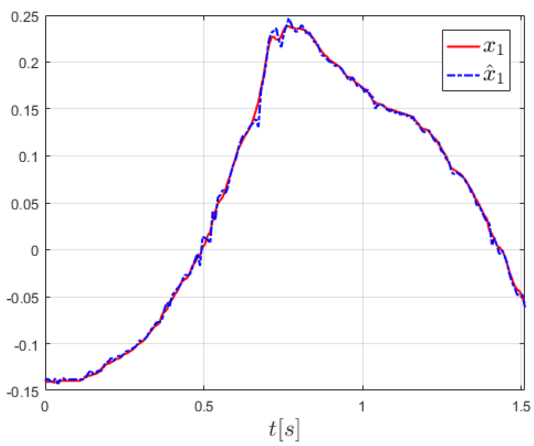

Fig. 6. Angle and estimated angle

with $\sigma = 24.8$, which can be verified *a posteriori* also on Fig 7.

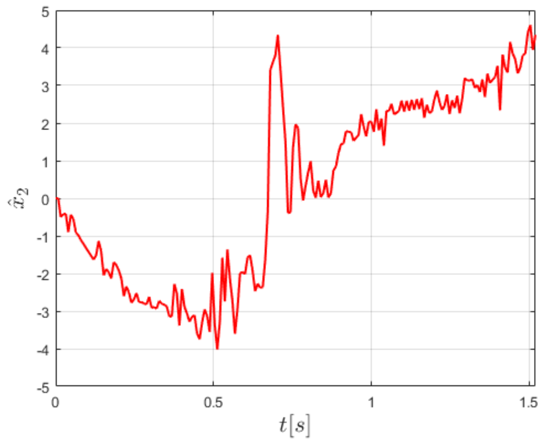

Fig. 7. Estimated angular velocity

VI. Conclusions and Future work

This paper presented a novel approach on observer design for discrete-time nonlinear systems with nonlinear consequents. To handle the measured state-nonlinearities TS fuzzy model was used, while the unmeasured-state terms were treated as slope-restricted nonlinearites. Sufficient conditions were formulated and tested on an inverted pendulum on a cart. An H_∞ performance index was also considered, and the conditions were formulated so that the effect of the disturbance was minimised. Both simulation and experiment provided good results.

There are many future directions, among which we will focus on extending this work to observer-based-controller design approach. On the other hand we plan to remove the single-input nonlinearity restriction to handle nonlinearities like $\sin(x_1 x_2)$.

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