Controller design for TS models using delayed non-quadratic Lyapunov functions

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***Abstract*—In the last few years, non-quadratic Lyapunov functions have been more and more frequently used in the analysis and controller design for Takagi-Sugeno fuzzy models. In this paper we developed relaxed conditions for controller design using nonquadratic Lyapunov functions and delayed controllers and give a general framework for the use of such Lyapunov functions. The two controller design methods developed in this framework outperform and generalize current state-of-the-art methods. The proposed methods are extended to robust and H_∞ control and α-sample variation.**

***Index Terms*—discrete-time Takagi-Sugeno models, controller design, non-quadratic Lyapunov functions, LMI.**

I. Introduction

Takagi-Sugeno (TS) fuzzy systems [1] are nonlinear, convex combinations of local linear models, and have the property that they are able to exactly represent a large class of nonlinear systems [2].

In order to analyze the stability or to design controllers and observers for a TS fuzzy model, the direct Lyapunov approach has been used. Stability conditions have been derived using quadratic Lyapunov functions [3], [4], [5], piecewise continuous Lyapunov functions [6], [7], and more recently, to reduce the conservativeness of the conditions, nonquadratic Lyapunov functions [8], [9], [10]. Other works try to introduce some properties of the membership function [11], or try to reduce the complexity of the LMI conditions. The stability or design conditions are generally derived in the form of linear matrix inequalities (LMIs).

Although also used for continuous-time TS models [12], [13], [14], [10], non-quadratic Lyapunov functions have shown a real improvement of the design conditions in the discrete-time case [8], [15], [16], [17]. It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework. A different type of improvement in the discrete case has been developed in [9], conditions being obtained by replacing the classical one sample variation of the Lyapunov function by its variation over several samples (α-sample variation).

More recently, by using Polya's theorem [18], [19] asymptotically necessary and sufficient (ANS) LMI conditions have been obtained for stability and stabilization in the sense of a chosen quadratic or nonquadratic Lyapunov function and control law. For linear-time invariant systems [20] investigated stability based on Polya's theorem and homogeneously polynomially parameter-dependent Lyapunov functions and ANS conditions in the sense of the membership function-dependent Lyapunov matrix have been obtained. [21] extended the results of [20] to TS models and homogeneous polynomially parameter-dependent non-parallel distributed compensation law, thus giving ANS stability conditions for both membership function-dependent model and membership function-dependent Lyapunov matrix. By increasing the complexity of the homogeneously polynomially parameter-dependent Lyapunov functions and the complexity of the homogeneous polynomially parameter dependent control laws, in theory any sufficiently smooth Lyapunov function and control law can be approximated. Thus [21] represents all Lyapunov functions and all control laws that are continuous in the membership functions. This issue deservers further study, as the conditions in many cases cannot be relaxed. Being ANS may not give a solution, due to computational intractability. The number of LMIs that have to be solved increase quickly, leading to numerical intractability [22]. Although slack matrices can be introduced to relax the conditions, these also increase the computational burden. Constructing a family of Lyapunov functions that covers the whole set of continuous Lyapunov functions can be done, but deciding which one is the most likely to be used to obtain good results is still an open problem. Moreover, a common assumption in all the results enumerated above is that the scheduling variables may not depend on the control input, in order to avoid solving implicit equations. Although this assumption is highly impractical as it means that the system has to be input-affine, it is necessary for the results in the literature.

With the considerations above, in this paper, we propose a general framework for using delayed non-quadratic Lyapunov functions for controller design. For discrete time TS models, in the nonquadratic framework, delayed controllers and observers have been proposed in [23]. The observer design method has been generalized further on in [24], but the controller design had the shortcoming of an increased number of LMIs. Thanks to the use of delayed Lyapunov functions and control laws, new possibilities for relaxing the derived conditions appear. While it is not our goal and we do not derive ANS conditions in this paper, using delayed Lyapunov functions and control laws, we show that the use of delay can lead to significant improvements. Finding a good structure of the control law reduces in a very important manner the conservatism of the results. The key point is finding a suitable solution that is compatible with actual solvers. Moreover, by using a delayed controller, the assumption that the scheduling variables must

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not depend on the current control input is no longer necessary, as solving the implicit equation is avoided. Thus, we present new possibilities for controller design based on the past states for a wider class of nonlinear systems. Furthermore, in the proposed framework, delayed systems can be easily handled. We also extend the results for robust control, H_∞ control and to α-sample variation, similar to [9].

The structure of the paper is as follows. Section II presents the notations used in this paper and the general form of the TS models, and motivates our work through a simple example. Section III develops the proposed conditions for controller design. The design methods are extended to robust control, α-sample variation and H_∞ control in Section IV. Section V concludes the paper.

II. Preliminaries

A. TS models

The discrete-time TS model considered in this paper for controller design is of the form

$$\boldsymbol{x}(k+1) = A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k) \tag{1}$$

where A_z denotes the convex sum $A_z = \sum_{i=1}^r h_i(\boldsymbol{z}(k))A_i$, A_i and B_i, $i = 1, 2, \ldots, r$ are the local matrices, r denotes the number of rules, k is the sample, $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state vector, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the control input, $\boldsymbol{z} \in \mathbb{R}^{n_z}$ is the scheduling vector. It is assumed that the scheduling variables $\boldsymbol{z}(k)$ are available at the sample k.

In what follows, we will make use of the following results:

Lemma 1. *[25] Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{m \times n_x}$ such that rank$(R) < n_x$. The two following expressions are equivalent:*

1) $\boldsymbol{x}^T Q\boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, R\boldsymbol{x} = 0\}$
2) $\exists M \in \mathbb{R}^{m\times n_x}$ *such that* $Q + MR + R^T M^T < 0$

Observer and controller design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{i=1}^r \sum_{j=1}^r h_i(\boldsymbol{z}(k))h_j(\boldsymbol{z}(k))\Gamma_{ij}\boldsymbol{x} < 0 \tag{2}$$

where Γ_{ij}, $i, j = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 2. *[26] The double-sum* (2) *is negative, if*

$$\begin{aligned}&\Gamma_{ii} < 0\\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i < j\end{aligned}$$

Lemma 3. *[27] The double-sum* (2) *is negative, if*

$$\begin{aligned}&\Gamma_{ii} < 0\\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i \neq j\end{aligned}$$

Property 1. *(Congruence) Given a matrix $P = P^T$ and a full column rank matrix Q it holds that*

$$P > 0 \;\Rightarrow\; QPQ^T > 0$$

Property 2. *Let A and B be matrices of appropriate dimensions and ranks, with $B = B^T > 0$. Then*

$$(A - B)^T B^{-1}(A - B) \geq 0 \Longleftrightarrow A^T B^{-1} A \geq A + A^T - B$$

Property 3. *[28] (Schur complement) Consider a matrix $M = M^T = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22}\end{pmatrix}$, with M_{11} and M_{22} being square matrices. Then*

$$\begin{aligned} M < 0 &\Leftrightarrow \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases} \\ &\Leftrightarrow \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}\end{aligned}$$

B. Motivation

For the sake of simplicity, in this section, we denote the convex sum $\sum_{i=1}^r h_i(\boldsymbol{z}(k))X_i$ as X_z, $X_z^{-1} = (\sum_{i=1}^r h_i(\boldsymbol{z}(k))X_i)^{-1}$, the subscript z meaning that the sum is evaluated at sample k. The subscript $z-$ stands for the sum being evaluated at sample $k-1$, e.g., $X_{z-} = \sum_{i=1}^r h_i(\boldsymbol{z}(k-1))X_i$, $z+$ means evaluation at sample $k+1$, e.g., $X_{z+} = \sum_{i=1}^r h_i(\boldsymbol{z}(k+1))X_i$, and multiple subscripts imply multiple sums, e.g., $X_{zz+} = \sum_{i=1}^r h_i(\boldsymbol{z}(k)) \sum_{j=1}^r h_j(\boldsymbol{z}(k+1))X_{ij}$.

In what follows, 0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry.

In [23], an observer design method based on the delayed Lyapunov function $V = \boldsymbol{e}^T P_{z-}\boldsymbol{e}$, $\boldsymbol{e}$ denoting the estimation error, has been proposed, that improved existing conditions, without increasing the number of LMIs to be solved. In what follows, we solve the dual problem, i.e., propose controller design conditions based on a delayed Lyapunov function that are able to improve the existing conditions without increasing the number of LMIs. This will serve as a motivating example for the general framework that will be presented in the following sections.

For the sake of the example, consider the TS model

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^r h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k))\\ &= A_z\boldsymbol{x}(k) + B_z\boldsymbol{u}(k)\end{aligned} \tag{3}$$

and the delayed controller similar to the one used in [23]

$$\boldsymbol{u}(k) = -F_{zz-}H_{zz-}^{-1}\boldsymbol{x}(k) \tag{4}$$

Using the Lyapunov function $V = \boldsymbol{x}^T P_{z-}^{-1}\boldsymbol{x}$, we have the difference

$$\begin{aligned}\Delta V_1 =& V(k+1) - V(k) = \\ =& \boldsymbol{x}(k+1)^T P_z^{-1}\boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T P_{z-}^{-1}\boldsymbol{x}(k)\\ =& \begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1}\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix}\end{aligned}$$

The closed-loop system dynamic is

$$\begin{pmatrix} A_z - B_zF_{zz-}H_{zz-}^{-1} & -I\end{pmatrix}\begin{pmatrix}\boldsymbol{x}(k)\\ \boldsymbol{x}(k+1)\end{pmatrix} = 0 \tag{5}$$

Using Lemma 1 with (5), $\Delta V_1 < 0$, if there exist $M \in \mathbb{R}^{2n_x \times n_x}$ so that

$$\begin{pmatrix} -P_{z-}^{-1} & 0 \\ 0 & P_z^{-1} \end{pmatrix} + M \begin{pmatrix} A_z - B_z F_{zz-} H_{zz-}^{-1} & -I \end{pmatrix} + (*) < 0$$

The implication 2. $\rightarrow$ 1. of Lemma 1 with the choice $M = \begin{pmatrix} 0 \\ P_z^{-1} \end{pmatrix}$ and congruence with $\begin{pmatrix} H_{zz-} & 0 \\ 0 & P_z \end{pmatrix}$ leads to

$$\begin{pmatrix} -H_{zz-}^T P_{z-}^{-1} H_{zz-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0$$

this being a sufficient condition. Using Property 2, we obtain

$$\begin{pmatrix} -H_{zz-} - H_{zz-}^T + P_{z-} & (*) \\ A_z H_{zz-} - B_z F_{zz-} & -P_z \end{pmatrix} < 0 \tag{6}$$

This result can be formulated as

Theorem 1. *The control law* (4) *asymptotically stabilizes the system* (1) *if there exist $P_z = P_z^T$, F_{zz-}, and H_{zz-}, so that* (6) *holds.*

Relaxed LMI conditions can easily be formulated using Lemmas 2 or 3, as follows.

Corollary 1. *The closed-loop system* (5) *is asymptotically stable, if there exist $P_i = P_i^T$, F_{ij}, and H_{ij}, $i, j = 1, 2, \ldots, r$ so that*

$$\Gamma_{ijk} + \Gamma_{jik} < 0$$

for $i, j, k = 1, 2, \ldots, r$, $i \le j$, or

$$\Gamma_{iik} < 0$$
$$\frac{2}{r-1}\Gamma_{iik} + \Gamma_{ijk} + \Gamma_{jik} < 0$$

for $i, j, k = 1, 2, \ldots, r$, where

$$\Gamma_{ijk} = \begin{pmatrix} -H_{jk} - H_{jk}^T + P_k & (*) \\ A_i H_{jk} - B_i F_{jk} & -P_i \end{pmatrix} < 0$$

The proof of the above corollary is straightforward.

Note that the conditions above are not equivalent to those in the literature, e.g., those in [8], which involve the negative definiteness of sums of the form

$$\begin{pmatrix} -P_z & (*) \\ A_z H_z - B_z F_z & -H_{z+} - H_{z+}^T + P_{z+} \end{pmatrix} < 0 \tag{7}$$

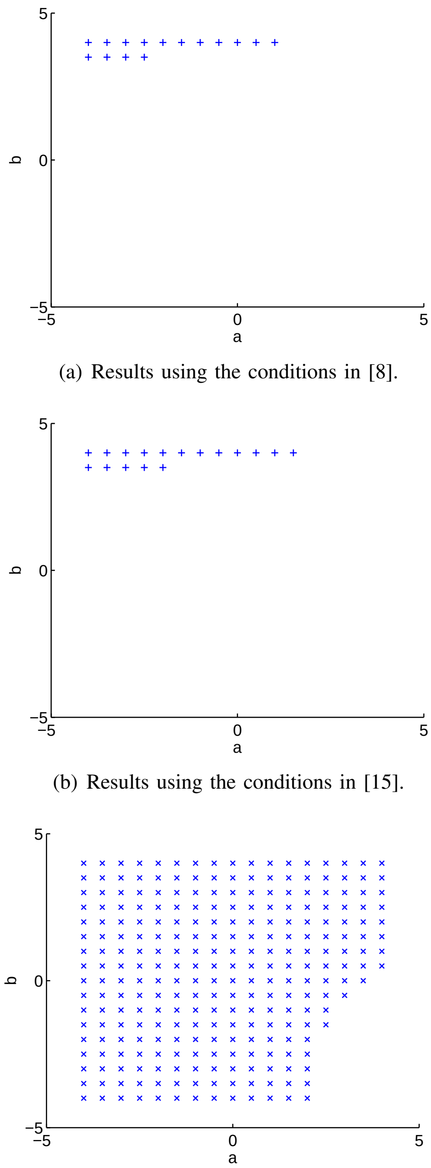
Remark: Both approaches lead to a triple sum, thus the number of conditions remains the same.

Consider the following example.

Example 1. *Consider the two-rule TS model having the local matrices*

$$A_1 = \begin{pmatrix} 2 & 0.04a + 6.9 \\ -1 & 0.03b - 2.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.03b - 2.9 \\ 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0.04a + 6.9 \\ -1 & 0.03b - 2.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

where a and b are real-valued parameters, $a, b \in [-4, 4]$. Using the conditions in [8], involving 3 sums, the values of a and b for which a solution can be found are presented in Figure 1(a). Using the conditions in [15], involving 4 sums, the values of a and b for which a solution can be found are presented in Figure 1(b). By using the conditions of Theorem 1, involving 3 sums, we obtain solutions for the values presented in Figure 1(c).



(a) Results using the conditions in [8].

(b) Results using the conditions in [15].

(c) Results using Theorem 1.

Fig. 1. Feasible solutions for Example 1.

Important remark: The Lyapunov function used, i.e., introducing P_{z-}^{-1} instead of P_z^{-1} has shown the possibility to cope with new controllers, i.e., $F_{zz-}H_{zz-}^{-1}$ instead of $F_z P_z^{-1}$ [8] or $F_z P_{zz}^{-1}$ [15]. This is crucial because in (6), $z-$ is present both in H_{zz-} and in P_{z-} and therefore helpful to reduce the conservatism as shown in the example. In comparison, approaches such as [8], [15], or [21] cannot introduce these extra degrees of freedom, as, because of the LMI formulation the controllers should depend on the future state $z+$, for instance $F_{zz+}P_z^{-1}$.

In view of this important remark, many new possibilities are now offered. Following [15], for example, we could introduce

P_{z-z-} or even generalize it as in [21]. Another possibility could be to use extra delayed states, for instance $z-1$, $z-2$, or mixed approaches. In order to propose such generalization we need to introduce some helpful notations.

C. Notations

For convenience, the following notations will be used in this paper.

Definition 1. (Multiple sum) We denote a multiple sum with n_P terms and delays evaluated at sample k of the form

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r} h_{i_1}(\boldsymbol{z}(k+d_1)) \sum_{i_2=1}^{r} h_{i_2}(\boldsymbol{z}(k+d_2)) \dots \sum_{i_{n_P}=1}^{r} h_{i_{n_P}}(\boldsymbol{z}(k+d_{n_P})) P_{i_1 i_2 \dots i_{n_P}}$$

where G_0^P is the multiset of delays $G_0^P = \{d_1, d_2, \dots, d_{n_P}\}$.

Definition 2. (Multiset of delays) G_0^P denotes the multiset containing the delays in the multiple sum involving P at sample k. G_α^P denotes the multiset containing the delays in the sum P at sample $k+\alpha$.

Definition 3. (Cardinality) The cardinality of a multiset G, $|G|$, is defined as the number of elements in G.

Definition 4. (Index set) The index set of a multiple sum $\mathbb{P}_G$ is $\mathbb{I}_G = \{i_j | i_j = 1, 2, \dots, r, \, j = 1, 2, \dots, |G|\}$, the set of all indices that appear in the sum. Note that these indices are directly related to the delays in G. An element $\boldsymbol{i} \in \mathbb{I}_G$ is a multiindex.

Definition 5. (Multiplicity) The multiplicity of an element x in a multiset G, $\mathbf{1}_G(x)$ denotes the number of times this element appears in the multiset G.

Definition 6. (Union) The union of two multisets G_A and G_B is $G_C = G_A \cup G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \max\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.

Definition 7. (Intersection) The intersection of two multisets G_A and G_B is $G_C = G_A \cap G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \min\{\mathbf{1}_{G_A}(x), \mathbf{1}_{G_B}(x)\}$.

Definition 8. (Sum) The sum of two multisets G_A and G_B is $G_C = G_A \oplus G_B$ such that $\forall x \in G_C$, $\mathbf{1}_{G_C}(x) = \mathbf{1}_{G_A}(x) + \mathbf{1}_{G_B}(x)$.

Definition 9. (Projection of an index) The projection of the index $\boldsymbol{i} \in \mathbb{I}_{G_A}$, to the multiset of delays G_B, $\mathrm{pr}_{G_B}^{\boldsymbol{i}}$ is the part of the index that corresponds to the delays in $G_A \cap G_B$.

Example 2. *Consider the multiple sum*

$$\mathbb{P}_{G_0^P} = \sum_{i_1=1}^{r} h_{i_1}(\boldsymbol{z}(k)) \sum_{i_2=1}^{r} h_{i_2}(\boldsymbol{z}(k)) \sum_{i_3=1}^{r} h_{i_3}(\boldsymbol{z}(k-1)) \sum_{i_4=1}^{r} h_{i_4}(\boldsymbol{z}(k-2)) P_{i_1 i_2 i_3 i_4}$$

The multiset of the delays at sample k, G_0^P is given by $G_0^P = \{0, 0, -1, -2\}$, while at sample $k+\alpha$ is $G_\alpha^P = \{\alpha, \alpha, \alpha-1, \alpha-2\}$.

The cardinality of G_0^P is $|G_0^P| = 4$, as $\mathbb{P}_{G_0^P}$ contains 4 sums. The multiplicity of the elements are $\mathbf{1}_{G_0^P}(0) = 2$, $\mathbf{1}_{G_0^P}(-1) = 1$, $\mathbf{1}_{G_0^P}(-2) = 1$. The multiplicity of an element that is not in G_0^P, e.g., 1 is $\mathbf{1}_{G_0^P}(1) = 0$.

The index set $\mathbb{I}_{G_0^P}$ of the multiple sum $\mathbb{P}_{G_0^P}$ is $\mathbb{I}_{G_0^P} = \{i_j | i_j = 1, 2, \dots, r, \, j = 1, 2, 3, 4\} = \{i_1 = 1, 2, \dots, r, \, i_2 = 1, 2, \dots, r, \, i_3 = 1, 2, \dots, r, \, i_4 = 1, 2, \dots, r\}$. An element of this set, e.g., $\boldsymbol{i} = 1234$, corresponding to $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, $i_4 = 4$, is a multiindex.

To illustrate the operations on multisets, consider two multisets $G_A = \{0, 0, -1, -2\}$, and $G_B = \{0, 0, -1, -1\}$. The union of these is $G_A \cup G_B = \{0, 0, -1, -1, -2\}$. The intersection is $G_A \cap G_B = \{0, 0, -1\}$. Their sum is $G_A \oplus G_B = \{0, 0, 0, 0-1, -1, -1, -2\}$.

The projection of the multiindex $\boldsymbol{i} = 1234$, $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$ to the multiset of delays $G_C = \{-1, -2\}$ is $pr_{G_C}^{\boldsymbol{i}} = 34$. Note that the projection of a multiindex is in general not unique. For instance, the projection of the element $\boldsymbol{i} = 1234 \in \mathbb{I}_{G_0^P}$ to $G_D = \{0, -1\}$ is either $pr_{G_C}^{\boldsymbol{i}} = 13$, i.e., $i_1 = 1$, $i_3 = 3$ or $pr_{G_C}^{\boldsymbol{i}} = 23$, i.e., $i_2 = 2$, $i_3 = 3$.

Remark: With the notations defined above, the system (1) can be written as

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A} \boldsymbol{x}(k) + \mathbb{B}_{G_0^B} \boldsymbol{u}(k) \tag{8}$$

with $G_0^A = G_0^B = \{0\}$. Moreover, one can easily use multiple sums involving delays in the system matrices, thus delayed systems can be easily handled. Although in what follows for the ease of notation we use the notation (8), in this paper we restrict ourselves to the classical TS system of the form (1).

III. Two controller design methods

In this paper we consider the problem of controller design for the system (8). The controller used for the system is of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F} \mathbb{H}_{G_0^H}^{-1} \boldsymbol{x}(k) \tag{9}$$

with $\mathbb{F}_{G_0^F}$ and $\mathbb{H}_{G_0^H}$ being multiple sums with delays given by G_0^F, $|G_0^F| = n_F$, and G_0^H, $|G_0^H| = n_H$, respectively. Note that G_0^F and G_0^H may not contain positive delays, since a positive delay refers to future scheduling variables, that are not available. At this point, the possible delays, i.e., the multisets G_0^F and G_0^H are not necessarily determined. Possible delays will be discussed in Section III-B. Note that this controller is a generalization of those used in [23], [21], [15], [8].

Using the controller (9) for the TS system (8), the closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A}\boldsymbol{x}(k) - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k) \tag{10}$$

A. Design conditions

To develop the design conditions, two different Lyapunov functions will be considered:

- **Case 1:** $V = \boldsymbol{x}(k)^T\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$, and $\mathbb{H}_{G_0^H}$ being the multisum used in the controller, and
- **Case 2:** $V = \boldsymbol{x}(k)^T\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$, with $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, for $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$.

In the remainder of this paper, whenever referring to Case 1 and Case 2, we refer to the two Lyapunov functions above.

For Case 1, the following result can be stated:

Theorem 2. *The closed-loop system* (10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr^{\boldsymbol{i}}_{G_j^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0,\,1$, *and* $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{G_0^F}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \end{pmatrix} < 0 \tag{11}$$

Remark: G_V above is simply the multiset containing all the delays in the multiple sum in (11).

Proof. *Consider the Lyapunov function* $V = \boldsymbol{x}(k)^T\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$, *with* $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, *for* $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. *The difference is*

$$\begin{aligned}\Delta V_1 =& V(k+1) - V(k) = \\ =& \boldsymbol{x}(k+1)^T\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\boldsymbol{x}(k+1) \\ &- \boldsymbol{x}(k)^T\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k) \\ =& \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\end{pmatrix}(*)\end{aligned}$$

The closed-loop system dynamics are

$$\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix}\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1)\end{pmatrix} = 0 \tag{12}$$

Using Lemma 1, $\Delta V_1 < 0$, *if there exists* $M \in \mathbb{R}^{2n_x \times n_x}$ *so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & 0 \\ 0 & \mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\end{pmatrix} + M\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I \end{pmatrix} + (*) < 0$$

In order to obtain a problem with LMI constraints encompassing the classical cases, a choice is:

$$M = \begin{pmatrix} 0 \\ \mathbb{H}_{G_1^H}^{-T}\end{pmatrix}$$

which leads to

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^{-T}\mathbb{P}_{G_0^P}\mathbb{H}_{G_0^H}^{-1} & (*) \\ \mathbb{H}_{G_1^H}^{-T}\mathbb{A}_{G_0^A} - \mathbb{H}_{G_1^H}^{-T}\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & \begin{pmatrix} -\mathbb{H}_{G_1^H}^{-T} - \mathbb{H}_{G_1^H}^{-1} \\ +\mathbb{H}_{G_1^H}^{-T}\mathbb{P}_{G_1^P}\mathbb{H}_{G_1^H}^{-1}\end{pmatrix}\end{pmatrix} < 0 \tag{13}$$

Applying to (13) *Property 1 with the full-rank matrix*

$$\begin{pmatrix} \mathbb{H}_{G_0^H}^T & 0 \\ 0 & \mathbb{H}_{G_1^H}^T\end{pmatrix}$$

gives directly the conditions (11). □

Case 2 leads to the conditions:

Theorem 3. *The closed-loop system* (10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $j = 0,\,1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{G_0^F}$, *and* $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr^{\boldsymbol{i}}_{G_0^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, *where* $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A)$ *so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P}\end{pmatrix} < 0 \tag{14}$$

Proof. *Consider the Lyapunov function* $V = \boldsymbol{x}(k)^T\mathbb{P}_{G_0^P}^{-1}\boldsymbol{x}(k)$, *with* $P_{\boldsymbol{i}} = P_{\boldsymbol{i}}^T > 0$, *for* $\boldsymbol{i} \in \mathbb{I}_{G_0^P}$, $|G_0^P| = n_P$. *The difference is*

$$\Delta V_1 = \begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1)\end{pmatrix}^T \begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1}\end{pmatrix}\begin{pmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k+1)\end{pmatrix}$$

Using Lemma 1 with (12), $\Delta V_1 < 0$, *if there exists* $M \in \mathbb{R}^{2n_x \times n_x}$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{G_0^P}^{-1} & 0 \\ 0 & \mathbb{P}_{G_1^P}^{-1}\end{pmatrix} + M\begin{pmatrix} \mathbb{A}_{G_0^A} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} & -I\end{pmatrix} + (*) < 0$$

Choosing $M = \begin{pmatrix} 0 \\ \mathbb{P}_{G_1^P}^{-1}\end{pmatrix}$ *and congruence with* $\begin{pmatrix} \mathbb{H}_{G_0^H} & 0 \\ 0 & \mathbb{P}_{G_1^P}\end{pmatrix}$ *leads to*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H}^T\mathbb{P}_{G_0^P}^{-1}\mathbb{H}_{G_0^H} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{P}_{G_1^P}\end{pmatrix} < 0$$

Using Property 2, we obtain directly (14). □

B. Discussion

First, we illustrate the use of the conditions (11) and (14), respectively, on the following example. Consider a two-rule fuzzy system

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k)) \\ &= \mathbb{A}_{G_0^A}\boldsymbol{x}(k) + \mathbb{B}_{G_0^B}\boldsymbol{u}(k)\end{aligned}$$

with $G_0^A = G_0^B = \{0\}$ for which a controller has to be designed and let $G_0^H = \{0, -1\}$, $G_0^F = \{0, -1\}$, $G_0^P =$

$\{-1,-1\}$, i.e.,

$$\mathbb{P}_{\{-1,-1\}} = \sum_{i=1}^{2}\sum_{j=1}^{2} h_i(\boldsymbol{z}(k-1))h_j(\boldsymbol{z}(k-1))P_{ij}$$

$$\mathbb{H}_{\{0,-1\}} = \sum_{i=1}^{2}\sum_{j=1}^{2} h_i(\boldsymbol{z}(k))h_j(\boldsymbol{z}(k-1))H_{ij}$$

$$\mathbb{F}_{\{0,-1\}} = \sum_{i=1}^{2}\sum_{j=1}^{2} h_i(\boldsymbol{z}(k))h_j(\boldsymbol{z}(k-1))F_{ij}$$

Then, the conditions (11) of Theorem 2 correspond to *there exist* P_{ij}*,* F_{ij}*,* H_{ij}*,* $i,j=1,\,2$ *so that*

$$\begin{pmatrix} -\mathbb{P}_{\{-1,-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0,-1\}} & -\mathbb{H}_{\{0,1\}} - \mathbb{H}^T_{\{0,1\}} + \mathbb{P}_{\{0,0\}} \end{pmatrix} < 0$$

or

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{i_4=1}^{2}\sum_{i_5=1}^{2} h_{i_1}(\boldsymbol{z}(k))h_{i_2}(\boldsymbol{z}(k))h_{i_3}(\boldsymbol{z}(k-1))$$
$$\cdot\, h_{i_4}(\boldsymbol{z}(k-1))h_{i_5}(\boldsymbol{z}(k+1))\cdot$$
$$\cdot \begin{pmatrix} -P_{i_3 i_4} & (*) \\ A_{i_1}H_{i_2 i_3} - B_{i_1}F_{i_2 i_3} & -H_{i_1 i_5} - H^T_{i_1 i_5} + P_{i_1 i_2} \end{pmatrix} < 0$$

while the conditions (14) of Theorem 3 correspond to *there exist* P_{ij}*,* F_{ij}*,* H_{ij}*,* $i,j=1,\,2$ *so that*

$$\begin{pmatrix} \mathbb{P}_{\{-1,-1\}} - \mathbb{H}_{\{0,-1\}} - \mathbb{H}^T_{\{0,-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0,-1\}} & -\mathbb{P}_{\{0,0\}} \end{pmatrix} < 0$$

or

$$\sum_{i_1=1}^{2}\sum_{i_2=1}^{2}\sum_{i_3=1}^{2}\sum_{i_4=1}^{2} h_{i_1}(\boldsymbol{z}(k))h_{i_2}(\boldsymbol{z}(k))h_{i_3}(\boldsymbol{z}(k-1))h_{i_4}(\boldsymbol{z}(k-1))\cdot$$
$$\cdot \begin{pmatrix} -H_{i_2 i_3} - H^T_{i_2 i_3} + P_{i_3 i_4} & (*) \\ A_{i_1}H_{i_2 i_3} - B_{i_1}F_{i_2 i_3} & -P_{i_1 i_2} \end{pmatrix} < 0$$

In what follows, we discuss how the multisets of delays should be chosen for a fixed number of sums such that relaxations (conservatism reduction approaches) can be used and the computational complexity, and implicitly the number of LMIs to be solved is reduced.

Let us see first some classical results. A quadratic Lyapunov functions and a PDC control law corresponds to $G_0^P = \emptyset$, $G_0^H = \emptyset$ and $G_0^F = \{0\}$. A nonquadratic approach such as the one in [15] corresponds to $G_0^P = \{0,0\}$, $G_1^P = \{1,1\}$, $G_0^H = G_0^F = \{0\}$. Recall that $G_0^A = G_0^B = \{0\}$. Thus, an important remark is that $G_0^P \cup (G_0^H \oplus G_0^A) \cup (G_0^F \oplus G_0^B) \cup G_1^P = G_V$ (Case 2) includes $\{0,0\}$ in order to apply known relaxations [19], [27]. Naturally, not every choice is adequate. For instance choosing $G_0^P = \emptyset$ and $G_0^F = G_0^H = \{-1,-1\}$ remains equivalent to a linear control, as the corresponding condition (Case 2) is

$$\begin{pmatrix} -H_{z_-z_-} - H^T_{z_-z_-} + P & (*) \\ A_z H_{z_-z_-} - B_z F_{z_-z_-} & -P \end{pmatrix} < 0$$

which has to be solved for every index and therefore each vertex becomes a solution.

Summarizing, G_V should include $\{0,0\}$ and more generally speaking the choice of the index sets has to favor in G_V multiple sums (at least 2) at the same samples. In this case the relaxations, for instance [27], [19], apply.

It can also be seen that the terms $\mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H}$ and $\mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}$, which appear in both cases, play a similar role. A convenient choice, without lack of generality, is $G_0^F = G_0^H$.

To illustrate the choice of the delays, consider now the simplest case, when $|G_0^P| = 1$, i.e., only one sum is used in P. Recall that in order to use relaxations, we choose G_0^F and G_0^H such that they contain $\{0\}$.

For Case 1, we have the inequality

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{H}_{\{1\}} - \mathbb{H}^T_{\{1\}} + \mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

which, independent of $|G_0^P|$ already contains 3 sums. By adding another index in G_0^H, the number of sums increases. Moreover, in order to keep this number of sums, $|G_0^P|$ has to be chosen as $|G_0^P| = \{0\}$. For an arbitrary cardinality of the multisets G_0^P and G_0^H, this generalizes to $G_0^P = \{0,\,0,\,\ldots,\,0\}$ and $G_0^F = G_0^H = \{0,\,0,\,\ldots,\,0\}$. Furthermore, if $|G_0^P| = |G_0^H| = n_P$, this choice reduces the number of sums in (11) to $2n_P+1$ and the number of LMIs to be solved (before relaxations) to r^{2n_p+1}. This can also be seen from $|G_V|$, which for this case is reduced to $G_V = \{0,\,\underbrace{0,\,0,\ldots,\,0}_{n_P},\,\underbrace{1,\,1,\ldots,\,1}_{n_P}\}$.

For Case 2, we have the inequality

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}^T_{\{0\}} + \mathbb{P}_{G_0^P} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{G_1^P} \end{pmatrix} < 0$$

which contains two sums. In order to have the same number of sums as in Case 1, G_0^P can be chosen either $\{0\}$ or $\{-1\}$, which would lead to

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}^T_{\{0\}} + \mathbb{P}_{\{0\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{\{1\}} \end{pmatrix} < 0$$

for $G_0^P = \{0\}$ or

$$\begin{pmatrix} -\mathbb{H}_{\{0\}} - \mathbb{H}^T_{\{0\}} + \mathbb{P}_{\{-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0\}} & -\mathbb{P}_{\{0\}} \end{pmatrix} < 0$$

for $G_0^P = \{-1\}$, respectively. It can be easily seen that in the second case, i.e., when $G_0^P = \{-1\}$, we can also add[1] another dimension to H and F, that provides more freedom, but without altering the number of sums in the condition. This choice will lead to the conditions

$$\begin{pmatrix} -\mathbb{H}_{\{0,-1\}} - \mathbb{H}^T_{\{0,-1\}} + \mathbb{P}_{\{-1\}} & (*) \\ \mathbb{A}_{\{0\}}\mathbb{H}_{\{0,-1\}} - \mathbb{B}_{\{0\}}\mathbb{F}_{\{0,-1\}} & -\mathbb{P}_{\{0\}} \end{pmatrix} < 0$$

For an arbitrary cardinality of the multisets G_0^P and G_0^H, this choice generalizes to $G_0^P = \{-1,\,-1,\,\ldots,\,-1\}$ and $G_0^F = G_0^H = \{0,\,0,\,\ldots,\,0,\,-1,\,\ldots,\,-1\}$. Moreover, if $|G_0^P| = |G_0^H| = 2n_P$, this choice reduces the number of sums in (11) to $2n_P+1$ and the number of LMIs to be solved (before relaxations) to r^{2n_p+1}. This can also be seen from $|G_V|$, which, similarly to Case 1, is reduced to $G_V = \{0,\,\underbrace{0,\,0,\ldots,\,0}_{n_P},\,\underbrace{-1,\,-1,\ldots,\,-1}_{n_P}\}$.

Taking in account these remarks, theoretically the more G_V grows, the best the results are. Of course a complexity issue

[1]Note that $\mathbb{H}_{\{0,1\}}$ cannot be used, as the premise variables are not known in advance.

due to the solvers limitation will constraint the growth of G_V and has to be discussed. The number of decision variables depends only on the number of sums used and for both cases is given by $r^{n_P} n_x(n_x+1)/2$ (the number of decision variables in $\mathbb{P}_{G_0^P}$) $+r^{n_H} n_x^2$ (the number of decision variables in $\mathbb{H}_{G_0^H}$) $+r^{n_F} n_x n_u$ (the number of decision variables in $\mathbb{F}_{G_0^F}$). However, the number of sums and consequently the number of LMI conditions to be solved differs.

The maximum number of sums in (11) (Case 1) is given by $|G_V| \leq 2n_P + n_F + 2n_H + 1$, which corresponds to the maximum[2] of $|G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_1^H|$ or pairwise non-overlapping sets of indices G_0^P, G_1^P, G_0^F, G_0^H, G_1^H, and $\{0\}$. For Case 2, the number of sums in (14) is given by $|G_V| \leq 2n_P + n_F + n_H + 1$, which in this case corresponds to the maximum of $|G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A)|$. The maximum number of sums indicates that for fixed G_0^H, G_0^F, and G_0^P, in general the conditions of Case 2 will lead to a smaller number of sums and consequently the number of LMIs to be solved.

In the analysis above, our aim was to reduce the computational complexity of the conditions. However, by increasing the number of sums in the terms (and implicitly the computational complexity), extra degrees of freedom are introduced and the obtained conditions become less conservative.

It should be noted that the conditions of Theorems 2 and 3, as they are stated, are nonlinear. Depending on how the LMIs are obtained, which relaxations are used, is Polya's theorem used, are slack variables introduced, etc., the computational complexity may quickly increase up to the point of numerical intractability. Moreover, depending on the relaxations used, the resulting LMIs for a specific application may be feasible or not. This is why we discuss the proposed conditions mainly in their nonlinear form, instead of the final LMI conditions that are usually reported in the literature.

Theorem 2 is a generalization of existing results and in a sense, a way to write them in a convenient general form. Theorem 3, by using delayed control, represents a new result that allows bringing new control laws that are not possible to use with the previous conditions. To show this, let us look to several results in the literature.

The results of [8] are recovered by using the Lyapunov function from Case 2 with $G_0^P = \{0\}$, $G_0^H = G_0^F = \{0\}$ and choosing $\mathbb{H}_{G^H} = \mathbb{P}_{G^P}$. Theorem 3 of [15] is obtained from Case 2, by choosing $G_0^P = \{0, 0\}$, and $G_0^H = G_0^F = \{0\}$. The controller design of [16] can be recovered from Theorem 2. The results in [29] (without the delay) are again a special case of Theorem 2. Theorem 1 of [30] is Theorem 4 for the choice $G_0^P = \{0, \ldots, 0\}$, $G_0^H = G_0^F = \{0, \ldots, 0\}$. Theorem 1 of [23] is obtained from Theorem 2 of this paper by choosing $G_0^P = \{-1\}$, and $G_0^H = G_0^F = \{0, -1\}$. The results of [17] (without the relaxations used on the sums) correspond to Theorem 2 applied for the special case of consecutive delays $G_0^P = \{-N+1, -N, \ldots, 0, 1\}$, $G_0^H = G_0^F = \{-N_f+1, -N_f, \ldots, 0\}$.

Note that the conditions of Theorems 2 and 3 are not equivalent, and although they generalize several conditions from the literature, they do not include each other. This will be illustrated on two examples. In order to obtain a fair comparison, the delays used are selected as $G_0^F = G_0^H = G_0^P = \{0\}$ for Case 1, and $G_0^F = G_0^H = \{0, -1\}$ and $G_0^P = \{-1\}$ for Case 2. This selection results in 3 sums both for (11) and (14). On the sums, the relaxation of [26] is used, and for solving the LMIs, the SeDuMi solver within the Yalmip [31] toolbox has been used.

[2] Assuming classical TS models, for which $G_0^A = G_0^B = \{0\}$.

Example 3. *Consider the two-rule fuzzy system*

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i \boldsymbol{x}(k) + B_i \boldsymbol{u}(k))$$

with

$$A_1 = \begin{pmatrix} -0.62 & 1.26 \\ 1.44 & -0.35 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1.04 & -0.26 \\ -0.66 & 0.45 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} -0.73 \\ 1.5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For this system, the conditions of Theorem 3 are unfeasible, while a result can be obtained using Theorem 2.

Example 4. *On the other hand, consider the two-rule fuzzy system*

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i \boldsymbol{x}(k) + B_i \boldsymbol{u}(k))$$

with

$$A_1 = \begin{pmatrix} 1.5 & 2.7 \\ -1.1 & 1.8 \end{pmatrix} \quad A_2 = \begin{pmatrix} -0.4 & -0.8 \\ 0.5 & -0.8 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} -0.55 \\ 0.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

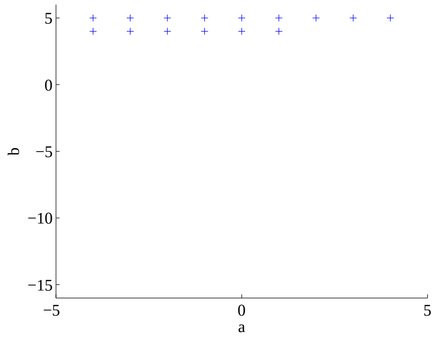
For this system, the conditions of Theorem 2 are unfeasible, but a result can be obtained using Theorem 3.

Example 5. *Let us now revisit Example 1. Recall that we consider the two-rule TS model having the local matrices*

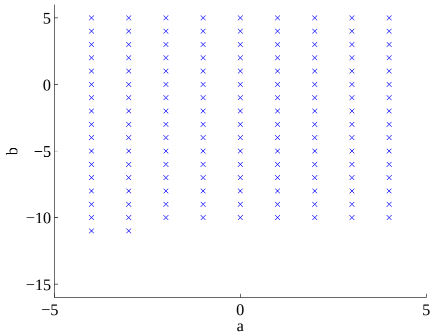
$$A_1 = \begin{pmatrix} 2 & 0.04a+6.9 \\ -1 & 0.03b-2.9 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.03b-2.9 \\ 1 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0.04a+6.9 \\ -1 & 0.03b-2.9 \end{pmatrix} \quad B_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

where a and b are real-valued parameters. For this case, we consider $a \in [-4, 4]$ and $b \in [-15, 5]$. We have already compared the conditions in [8], and those of Theorem 1, which both involved 3 sums. Generalizing the conditions to 4 sums, we have $G_0^P = \{0, 0\}$ and $G_0^F = G_0^H = \{0\}$ for Theorem 2 and $G_0^P = \{-1, -1\}$ and $G_0^F = G_0^H = \{0, -1, -1\}$ for Theorem 3. The values of a and b for which we obtain solutions are presented in Figures 2(a) and 2(b).

Let us now compare the conditions proposed in this paper to those in the literature. To consider the simplest case, the relaxation of [26] is used on all the possible sums, but slack (auxiliary) variables are not used.



(a) Results using Theorem 2.



(b) Results using Theorem 3.

Fig. 2. Feasible solutions when using 4 sums for Example 1.

Example 6. *Consider the two-rule TS model [8], [15], [21], [22] having the local matrices*

$$A_1 = \begin{pmatrix} 1 & -b \\ -1 & -0.5 \end{pmatrix} \quad B_1 = \begin{pmatrix} 5+b \\ 2b \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & b \\ -1 & -0.5 \end{pmatrix} \quad B_2 = \begin{pmatrix} 5-b \\ -2b \end{pmatrix} \tag{15}$$

where b is a real-valued parameter. This system has been used for illustrating the results in the literature. The goal is to be able to stabilize the system for the parameter b having a value as large as possible. In the literature the following values have been reported: [15], $b = 1.7669$, [21], $b = 1.82$, [30], $b = 1.82$, [22], $b = 1.8106$. According to [22], to obtain $b = 1.8106$ it took $3\frac{1}{3}$min or $1\frac{4}{5}$h, depending on which theorem and relaxation they used. No value greater than $b = 1.83$ has been reported in the literature.

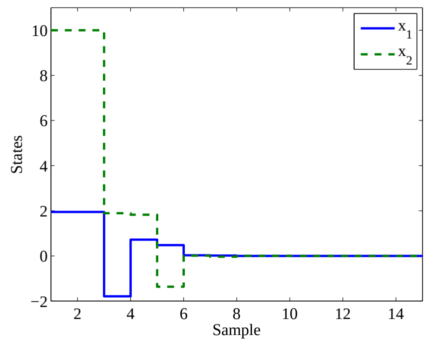
To illustrate the advantage of using delayed Lyapunov function and delayed controllers, we use Theorem 3 and the following delays: $G_0^P = \{-1, -1, -1\}$, $G_0^H = G_0^F = \{0, 0, -1, -1, -1\}$. This means that the Lyapunov "matrix" is a triple sum, with each depending on the past scheduling instead of the current one, while the control gains consist of five sums, out of which three depend on the past scheduling and two on the current one, i.e., we use

$$\mathbb{P}_{\{-1,\,-1\,-1\}} = \sum_{i_1=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_2=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_3=1}^{r}(\boldsymbol{z}(k-1))P_{i_1 i_2 i_3}$$

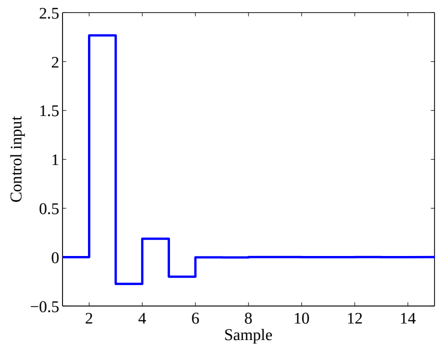
$$\mathbb{H}_{\{0,\,0,\,-1,\,-1\,-1\}} = \sum_{i_1=1}^{r}(\boldsymbol{z}(k)) \sum_{i_2=1}^{r}(\boldsymbol{z}(k)) \sum_{i_3=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_4=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_5=1}^{r}(\boldsymbol{z}(k-1))H_{i_1 i_2 i_3 i_4 i_5}$$

$$\mathbb{F}_{\{0,\,0,\,-1,\,-1\,-1\}} = \sum_{i_1=1}^{r}(\boldsymbol{z}(k)) \sum_{i_2=1}^{r}(\boldsymbol{z}(k)) \sum_{i_3=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_4=1}^{r}(\boldsymbol{z}(k-1)) \sum_{i_5=1}^{r}(\boldsymbol{z}(k-1))F_{i_1 i_2 i_3 i_4 i_5}$$

With these delays, the conditions of Theorem 3, using Lemma 2 to obtain the LMIs result in 16 LMIs to be solved. Solving the LMIs takes 0.61sec[3]. With these parameters, the system (15) can be controlled when $b = 1.95$. For the choice $\boldsymbol{x}(0) = [1.95,\ 10]^T$, the closed-loop system trajectories and the control input are given in Figures 3(a) and 3(b). This is a major improvement wrt. the results reported in the literature.



(a) Closed-loop system states.



(b) Control input.

Fig. 3. Closed-loop system trajectories for $b = 1.95$ for Example 6.

Let us now compare our results to those in the literature based on the number of sums in the conditions and the largest

[3]For solving LMIs a laptop with processor Intel Core i5 @ 2.6GHz and with 8 GB RAM has been used.

value of b being reported. This – not taking into account what relaxation is used – gives a measure of complexity of the conditions.

The conditions of Theorem 3 in [15] can be written a special case of Theorem 3, with $G_0^P = \{0, 0\}$ and $G_0^F = G_0^H = \{0\}$. The number of sums is 4, and the maximum value of b for which the LMIs are feasible is $b = 1.547$. The conditions presented in [8] (special case of Theorem 2, with $G_0^P = G_0^F = G_0^H = \{0\}$) are feasible up to $b = 1.539$, although they only involve 3 sums. Applying Theorem 2 of [17] with $G_0^P = \{-1, 0, 1\}$ and $G_0^H = G_0^F = \{-1, 0\}$ using the relaxation of [26] and without slack variables, $b = 1.565$ is obtained, but the conditions involve 5 sums. For the choice $G_0^P = \{0, 1\}$ and $G_0^H = G_0^F = \{0\}$, which involves only 4 sums, the maximum b is $b = 1.54$.

Consider now the conditions proposed in this paper. For Theorem 2, to obtain 4 sums, one can choose e.g., $G_0^P = \{0, 0\}$ and $G_0^F = G_0^H = \{0\}$. With this choice, the maximum b obtained is $b = 1.547$, i.e., the same as in [15]. For Theorem 3, by choosing $G_0^P = \{-1\}$ and $G_0^F = G_0^H = \{0, 0, -1\}$, we obtain $b = 1.589$. Moreover, by choosing $G_0^P = \{-1\}$ and $G_0^F = G_0^H = \{0, -1\}$ – only 3 sums – we get $b = 1.553$. The results are summarized in Table I.

TABLE I
COMPARISON OF RESULTS

Method	Nr. of sums	Maximum b
[8]	3	1.539
Theorem 3, this paper	3	1.553
Theorem 3 in [15]	4	1.547
Theorem 2 of [17]	4	1.54
Theorem 2, this paper	4	1.56
Theorem 3, this paper	4	1.69
Theorem 2 of [17]	5	1.565
Theorem 2, this paper	5	1.69
Theorem 3, this paper	5	1.78
Theorem 3, this paper	6	1.95

Let us also compare the results from the literature from the point of view of number of LMIs to be solved and the largest b reported. According to [21], for $p = 2$, $d = 1$, $d_+ = 1$, (20 LMIs to solve) the result obtained is $b = 1.68$; for $p = d = d_+ = 2$, 30 LMIs, $b = 1.71$; $p = d = d_+ = 3$, 56 LMIs, $b = 1.74$. The results of our Theorem 2 are comparable to these results, as the controllers are similar. However, using Theorem 3 with 4 sums – actually solving 8 LMIs by applying the relaxation of [26] –, results in $b = 1.69$; 5 sums, 12 LMIs to solve, $b = 1.78$; 6 sums, 16 LMIs to solve, $b = 1.95$. Consequently, in our approach the number of LMIs is much reduced.

IV. Extensions

In what follows, the results presented in Section III-A are extended to robust controllers, α-sample variation and H_∞ control. Since these extensions are quite classical given the framework we propose, they are presented without proofs.

A. Robust controllers

First, let us extend the conditions of Section III-A for the case when the system is described by

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} + \Delta A)\boldsymbol{x}(k) + (\mathbb{B}_{G_0^B} + \Delta B)\boldsymbol{u}(k) \tag{16}$$

i.e., the local matrices are uncertain. The uncertainties considered are of the form $\Delta A = \mathbb{D}_{G_{0,a}^D}\Delta_a\mathbb{E}_{G_{0,a}^E}$ and $\Delta B = \mathbb{D}_{G_{0,b}^D}\Delta_b\mathbb{E}_{G_{0,b}^E}$, with $\Delta_a^T\Delta_a < I$, and $\Delta_b^T\Delta_b < I$.

The controller is of the form

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

and the closed-loop system can be expressed as

$$\boldsymbol{x}(k+1) = (\mathbb{A}_{G_0^A} + \Delta A - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1} - \Delta B\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1})\boldsymbol{x}(k) \tag{17}$$

Then, the following results can be stated:

Theorem 4. *The closed-loop system* (17) *is asymptotically stable, if there exist $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, and $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr^{\boldsymbol{i}}_{G_j^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{G_0^F}$, $S_{\boldsymbol{i}_{0,a}^S} = S_{\boldsymbol{i}_{0,a}^S}^T > 0$, $\boldsymbol{i}_{0,a}^S = pr^{\boldsymbol{i}}_{G_{0,a}^S}$, and $S_{\boldsymbol{i}_{0,b}^S} = S_{\boldsymbol{i}_{0,b}^S}^T > 0$, $\boldsymbol{i}_0^{S_b} = pr^{\boldsymbol{i}}_{G_{0,b}^S}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus (G_0^B \cup G_{0,b}^E)) \cup (G_0^H \oplus (G_0^A \cup G_{0,a}^E)) \cup G_1^H \cup (G_{0,a}^S \oplus G_{0,a}^D \oplus G_{0,a}^D) \cup (G_{0,b}^S \oplus G_{0,b}^D \oplus G_{0,b}^D)$ so that* (18) *holds.*

Theorem 5. *The closed-loop system* (17) *is asymptotically stable, if there exist $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, and $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr^{\boldsymbol{i}}_{G_0^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr^{\boldsymbol{i}}_{G_0^F}$, $S_{\boldsymbol{i}_{0,a}^S} = S_{\boldsymbol{i}_{0,a}^S}^T > 0$, $\boldsymbol{i}_{0,a}^S = pr^{\boldsymbol{i}}_{G_{0,a}^S}$, and $S_{\boldsymbol{i}_{0,b}^S} = S_{\boldsymbol{i}_{0,b}^S}^T > 0$, $\boldsymbol{i}_0^{S_b} = pr^{\boldsymbol{i}}_{G_{0,b}^S}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus (G_0^B \cup G_{0,b}^E)) \cup (G_0^H \oplus (G_0^A \cup G_{0,a}^E)) \cup (G_{0,a}^S \oplus G_{0,a}^D \oplus G_{0,a}^D) \cup (G_{0,b}^S \oplus G_{0,b}^D \oplus G_{0,b}^D)$ so that* (19) *holds.*

In what follows, we illustrate Theorems 4 and 5 on an example.

Example 7. *Consider the uncertain two-rule fuzzy system*

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))((A_i + \Delta A)\boldsymbol{x}(k) + B_i\boldsymbol{u}(k))$$

with

$$\Delta A = D_a\Delta_a E_a \qquad \Delta_a^T\Delta_a \leq I$$

$$A_1 = \begin{pmatrix} 1 & -1.65 \\ -1 & -0.5 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & 1.65 \\ -1 & -0.5 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 6.65 \\ 3.3 \end{pmatrix} \quad B_2 = \begin{pmatrix} 3.35 \\ -3.3 \end{pmatrix}$$

$$D_a = \begin{pmatrix} -0.0465 & 0.07 \\ 0.0371 & 0.21 \end{pmatrix} \qquad E_a = \begin{pmatrix} -0.1357 & 0.10 \\ -0.1 & -0.039 \end{pmatrix}$$

Using Theorem 4, with a single sum both in the Lyapunov function and the controller gains, i.e., $G_0^P = G_0^H = G_0^F = \{0\}$, we obtain the matrices

$$H_1 = \begin{pmatrix} 0.21 & -0.008 \\ 0.044 & 0.74 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0.29 & -0.008 \\ -0.027 & 0.51 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 0.0123 & -0.19 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0.068 & 0.21 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0.29 & 0.029 \\ 0.028 & 0.68 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.34 & -0.04 \\ -0.04 & 0.44 \end{pmatrix}$$

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ \mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H} & -\mathbb{S}_{G_{0,a}^S} & (*) & (*) \\ \mathbb{E}_{G_{0,b}^E}\mathbb{F}_{G_0^F} & 0 & -\mathbb{S}_{G_{0,b}^S} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & 0 & 0 & \begin{pmatrix} -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} \\ +\mathbb{D}_{G_{0,a}^D}^T\mathbb{S}_{G_{0,a}^S}\mathbb{D}_{G_{0,a}^D} + \mathbb{D}_{G_{0,b}^D}^T\mathbb{S}_{G_{0,b}^S}\mathbb{D}_{G_{0,b}^D} \end{pmatrix} \end{pmatrix} < 0 \tag{18}$$

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ \mathbb{E}_{G_{0,a}^E}\mathbb{H}_{G_0^H} & -\mathbb{S}_{G_{0,a}^S} & (*) & (*) \\ \mathbb{E}_{G_{0,b}^E}\mathbb{F}_{G_0^F} & 0 & -\mathbb{S}_{G_{0,b}^S} & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & 0 & 0 & \begin{pmatrix} -\mathbb{P}_{G_1^P} + \mathbb{D}_{G_{0,a}^D}^T\mathbb{S}_{G_{0,a}^S}\mathbb{D}_{G_{0,a}^D} \\ +\mathbb{D}_{G_{0,b}^D}^T\mathbb{S}_{G_{0,b}^S}\mathbb{D}_{G_{0,b}^D} \end{pmatrix} \end{pmatrix} < 0 \tag{19}$$

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & \dots & 0 \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T & \dots & 0 \\ 0 & \mathbb{A}_{G_1^A}\mathbb{H}_{G_1^H} - \mathbb{B}_{G_1^B}\mathbb{F}_{G_1^F} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \mathbb{P}_{G_\alpha^P} - \mathbb{H}_{G_\alpha^H}^T - \mathbb{H}_{G_\alpha^H} \end{pmatrix} < 0 \tag{20}$$

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) & \dots & 0 & 0 \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T & \dots & 0 & 0 \\ 0 & \mathbb{A}_{G_1^A}\mathbb{H}_{G_1^H} - \mathbb{B}_{G_1^B}\mathbb{F}_{G_1^F} & \dots & 0 & \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -\mathbb{H}_{G_{\alpha-1}^H} - \mathbb{H}_{G_{\alpha-1}^H}^T & 0 \\ 0 & 0 & \dots & \mathbb{A}_{G_{\alpha-1}^A}\mathbb{H}_{G_{\alpha-1}^H} - \mathbb{B}_{G_{\alpha-1}^B}\mathbb{F}_{G_{\alpha-1}^F} & -\mathbb{P}_{G_\alpha^P} \end{pmatrix} < 0 \tag{21}$$

The conditions of Theorem 4 involved 3 sums, and to write LMI conditions, Lemma 3 has been used.

The delays for Theorem 5 are chosen such that also 3 sums are obtained, i.e., $G_0^P = \{-1\}$, $G_0^H = G_0^F = \{0, -1\}$, *and to write LMI conditions, Lemma 3 has been used. We obtain*

$$H_{11} = \begin{pmatrix} 0.26 & 0.03 \\ -0.087 & 0.68 \end{pmatrix} \qquad H_{12} = \begin{pmatrix} 0.25 & 0.025 \\ 0.12 & 0.70 \end{pmatrix}$$

$$H_{21} = \begin{pmatrix} 0.29 & -0.11 \\ -0.069 & 0.56 \end{pmatrix} \qquad H_{22} = \begin{pmatrix} 0.23 & 0.06 \\ 0.018 & 0.52 \end{pmatrix}$$

$$F_{11} = \begin{pmatrix} 0.04 & -0.17 \end{pmatrix} \qquad F_{12} = \begin{pmatrix} -0.01 & -0.18 \end{pmatrix}$$

$$F_{21} = \begin{pmatrix} 0.05 & 0.18 \end{pmatrix} \qquad F_{22} = \begin{pmatrix} 0.07 & 0.22 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0.20 & -0.10 \\ -0.10 & 0.67 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.19 & 0.09 \\ 0.09 & 0.63 \end{pmatrix}$$

B. α-sample variation

In this section, we extend the results obtained in Section III-A using α-sample variation of the Lyapunov function [9].

Recall that by using the controller (repeated here for convenience)

$$\boldsymbol{u}(k) = -\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

the closed-loop system is given by

$$\boldsymbol{x}(k+1) = \mathbb{A}_{G_0^A}\boldsymbol{x}(k) - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k)$$

For Case 1, the following result can be stated:

Theorem 6. *The closed-loop system* (10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = pr^{\boldsymbol{i}}_{G_j^F}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr^{\boldsymbol{i}}_{G_j^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1, 2, \ldots, \alpha$, *where* $G_V = G_0^P \cup G_\alpha^P \cup \bigcup_{i=0}^{\alpha-1}(G_i^F \oplus G_i^B) \cup \bigcup_{i=0}^{\alpha-1}(G_i^H \oplus G_i^A) \cup G_\alpha^H$ *so that* (20) *holds.*

Remark: Note that again, G_V denotes the multiset of all the delays that appear in the sum in (20). The terms $\bigcup_{i=0}^{\alpha-1}(G_i^H \oplus G_i^A)$ and $\bigcup_{i=0}^{\alpha-1}(G_i^F \oplus G_i^B)$ are actually the delays that appear in the terms $\mathbb{A}_{G_i^A}\mathbb{H}_{G_i^H}$ and $\mathbb{B}_{G_i^B}\mathbb{F}_{G_i^F}$, $i = 1, 2, \ldots, \alpha - 1$.

For Case 2, we have the following conditions.

Theorem 7. *The closed-loop system* (10) *is asymptotically stable, if there exist* $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr^{\boldsymbol{i}}_{G_j^P}$, $F_{\boldsymbol{i}_j^F}$, $\boldsymbol{i}_j^F = pr^{\boldsymbol{i}}_{G_j^F}$, *and* $H_{\boldsymbol{i}_j^H}$, $\boldsymbol{i}_j^H = pr^{\boldsymbol{i}}_{G_j^H}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, $j = 0, 1, 2, \ldots, \alpha$, *where* $G_V = G_0^P \cup G_\alpha^P \cup \bigcup_{i=0}^{\alpha-1}(G_i^F \oplus G_i^B) \cup \bigcup_{i=0}^{\alpha-1}(G_i^H \oplus G_i^A) \cup G_\alpha^H$ *so that* (21) *holds.*

Remark: Similarly to Section III-A, depending on the exact sets of indices used, G_0^P, G_0^F, and G_0^H, relaxations such as [26], [3], [27], [19] can be used.

C. H_∞-control

In this section, we consider H_∞ control using the controller (10). Consider then the system expressed as:

$$\begin{aligned} \boldsymbol{x}(k+1) &= \mathbb{A}_{G_0^A}\boldsymbol{x}(k) + \mathbb{B}_{G_0^B}\boldsymbol{u}(k) + \mathbb{E}_{G_0^E}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= \mathbb{C}_{G_0^C}\boldsymbol{x}(k) + \mathbb{D}_{G_0^D}\boldsymbol{u}(k) + \mathbb{K}_{G_0^K}\boldsymbol{w}(k) \end{aligned} \tag{22}$$

$$\begin{pmatrix} -\mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ 0 & -\gamma I & (*) & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{H}_{G_1^H} - \mathbb{H}_{G_1^H}^T + \mathbb{P}_{G_1^P} & (*) \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\gamma I \end{pmatrix} < 0 \tag{25}$$

The closed-loop system is given by

$$\begin{aligned} \boldsymbol{x}(k+1) &= \mathbb{A}_{G_0^A}\boldsymbol{x}(k) - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k) + \mathbb{E}_{G_0^E}\boldsymbol{w}(k) \\ \boldsymbol{y}(k) &= \mathbb{C}_{G_0^C}\boldsymbol{x}(k) - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F}\mathbb{H}_{G_0^H}^{-1}\boldsymbol{x}(k) + \mathbb{K}_{G_0^K}\boldsymbol{w}(k) \end{aligned} \tag{23}$$

For Case 1, the following result can be stated:

Theorem 8. *The closed-loop system* (23) *is asymptotically stable, and the attenuation is at least γ if there exist $\gamma > 0$, $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, and $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr_{G_0^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_0^K \cup G_0^E$ so that* (25) *holds.*

For Case 2, we have

Theorem 9. *The closed-loop system* (23) *is asymptotically stable, and the attenuation is at least γ if there exist $\gamma > 0$, $P_{\boldsymbol{i}_j^P} = P_{\boldsymbol{i}_j^P}^T$, $\boldsymbol{i}_j^P = pr_{G_j^P}^{\boldsymbol{i}}$, $j = 0, 1$, $F_{\boldsymbol{i}_0^F}$, $\boldsymbol{i}_0^F = pr_{G_0^F}^{\boldsymbol{i}}$, and $H_{\boldsymbol{i}_0^H}$, $\boldsymbol{i}_0^H = pr_{G_0^H}^{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{I}_{G_V}$, where $G_V = G_0^P \cup G_1^P \cup (G_0^F \oplus G_0^B) \cup (G_0^H \oplus G_0^A) \cup G_0^K \cup G_0^E$ so that*

$$\begin{pmatrix} -\mathbb{H}_{G_0^H} - \mathbb{H}_{G_0^H}^T + \mathbb{P}_{G_0^P} & (*) & (*) & (*) \\ 0 & -\gamma I & (*) & (*) \\ \mathbb{A}_{G_0^A}\mathbb{H}_{G_0^H} - \mathbb{B}_{G_0^B}\mathbb{F}_{G_0^F} & \mathbb{E}_{G_0^E} & -\mathbb{P}_{G_1^P} & (*) \\ \mathbb{C}_{G_0^C}\mathbb{H}_{G_0^H} - \mathbb{D}_{G_0^D}\mathbb{F}_{G_0^F} & \mathbb{K}_{G_0^K} & 0 & -\gamma I \end{pmatrix} < 0 \tag{24}$$

In what follows, we illustrate Theorems 8 and 9 on an example.

Example 8. *Consider the two-rule fuzzy system*

$$\begin{aligned} \boldsymbol{x}(k+1) &= \sum_{i=1}^{2} h_i(\boldsymbol{z}(k))(A_i\boldsymbol{x}(k) + B_i\boldsymbol{u}(k) + E\boldsymbol{w}(k)) \\ \boldsymbol{y}(k) &= \boldsymbol{x}(k) \end{aligned}$$

with

$$A_1 = \begin{pmatrix} 1 & -1.65 \\ -1 & -0.5 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 1.65 \\ -1 & -0.5 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 6.65 \\ 3.3 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 3.35 \\ -3.3 \end{pmatrix}$$
$$E_a = \begin{pmatrix} -0.1357 & 0.10 \\ -0.1 & -0.039 \end{pmatrix}$$

Using Theorem 8, with a single sum both in the Lyapunov function and the controller gains, i.e., $G_0^P = G_0^H = G_0^F = \{0\}$, we obtain an attenuation $\gamma = 1.71$ and the matrices

$$H_1 = \begin{pmatrix} 0.06 & -0.01 \\ -0.005 & 0.32 \end{pmatrix} \qquad H_2 = \begin{pmatrix} 0.02 & 0.037 \\ -0.01 & 0.17 \end{pmatrix}$$
$$F_1 = \begin{pmatrix} 0.014 & -0.07 \end{pmatrix} \qquad F_2 = \begin{pmatrix} 0.002 & 0.08 \end{pmatrix}$$
$$P_1 = \begin{pmatrix} 0.08 & -0.03 \\ -0.03 & 0.42 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.001 & -0.008 \\ -0.01 & 0.18 \end{pmatrix}$$

The conditions of Theorem 8 involved 3 sums, and to write LMI conditions, Lemma 3 has been used.

The delays for Theorem 9 have been chosen such that also 3 sums are obtained, i.e., $G_0^P = \{-1\}$, $G_0^H = G_0^F = \{0, -1\}$, and to write LMI conditions, Lemma 3 has been used. We obtain an attenuation $\gamma = 1.37$ – better than that obtained using Theorem 8 – and the matrices:

$$H_{11} = \begin{pmatrix} 0.07 & -0.006 \\ -0.02 & 0.27 \end{pmatrix} \qquad H_{12} = \begin{pmatrix} 0.08 & -0.04 \\ -0.007 & 0.32 \end{pmatrix}$$
$$H_{21} = \begin{pmatrix} 0.17 & -0.04 \\ -0.04 & 0.17 \end{pmatrix} \qquad H_{22} = \begin{pmatrix} 0.01 & 0.04 \\ 0.003 & 0.17 \end{pmatrix}$$
$$F_{11} = \begin{pmatrix} 0.01 & -0.07 \end{pmatrix} \qquad F_{12} = \begin{pmatrix} 0.005 & -0.08 \end{pmatrix}$$
$$F_{21} = \begin{pmatrix} 0.03 & 0.06 \end{pmatrix} \qquad F_{22} = \begin{pmatrix} 0.006 & 0.09 \end{pmatrix}$$
$$P_1 = \begin{pmatrix} 0.06 & -0.05 \\ -0.05 & 0.17 \end{pmatrix} \qquad P_2 = \begin{pmatrix} 0.03 & 0.05 \\ 0.05 & 0.17 \end{pmatrix}$$

V. Conclusions

This paper presented a general framework for the design of nonquadratic controllers for TS fuzzy models. Two methods have been proposed, depending on the Lyapunov function used. It has been shown that the proposed controllers include controllers reported in the recent literature. In our future research we will analyze which method should be used for what type of problems. The design methods have also been extended to robust control, α-sample variation, and H_∞ control.

In our future research we will develop controller design methods for delayed systems, which in this framework can easily be addressed. We also aim to develop asymptotically necessary and sufficient conditions for controller design.

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