# Stability Bounds for Fuzzy Estimation and Control

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**Abstract:** A large class of nonlinear systems can be well approximated by Takagi-Sugeno fuzzy models, for which methods and algorithms have been developed to analyze their stability and to design observers and controllers. However, results obtained for Takagi-Sugeno fuzzy models are in general not directly applicable to the original nonlinear system. In this paper, we investigate what conclusions can be drawn and what guarantees can be expected when an observer or a state feedback controller is designed based on an approximate fuzzy model and applied to the original nonlinear system. We also investigate the case when an observer-based controller is designed for an approximate model and then applied to the original nonlinear system. In particular, we consider that the scheduling vector used in the membership functions of the observer depends on the states that have to be estimated. The results are illustrated using simulation examples.

*Keywords:* Fuzzy systems, fuzzy approximations, Lyapunov stability, boundedness.

## 1. INTRODUCTION

A large class of nonlinear functions can be exactly represented or accurately approximated by Takagi-Sugeno (TS) fuzzy models (Takagi and Sugeno, 1985). The TS fuzzy model consists of a rule base. The rule antecedents partition a subset of the system's variables into fuzzy regions, while the consequent of each rule is a linear or affine model, valid locally in the corresponding region.

A well-known method to obtain an exact TS fuzzy representation of a nonlinear system is the sector nonlinearity approach (Ohtake et al., 2001). However, when using this method, the observability and controllability of the local models is not guaranteed, even when the original nonlinear system is observable and controllable. Although for fuzzy models well-established methods exist to analyze their stability or to design observers and controllers, these cannot be used if the local models are not stable, observable, or controllable, respectively.

In this paper we consider fuzzy models that retain observability and controllability in their local models, even though they only approximate the nonlinear system. Several methods exist to construct TS models such that they approximate a given nonlinear model to an arbitrary degree of accuracy (Fantuzzi and Rovatti, 1996; Johansen et al., 2000; Kiriakidis, 2007). In this case, since the fuzzy model only approximates the original nonlinear system, when the analysis or design concerns the fuzzy model, the results may not directly hold true for the nonlinear system. For instance, the observers or controllers designed for the fuzzy model are in general not guaranteed to perform as expected for the nonlinear system.

For a class of nonlinear systems, when using control based on TS fuzzy models, this shortcoming has been circumvented by the use of robust controllers. Robust fuzzy control has attracted increased research interest in the last decade. Results range from fuzzy control of nonlinear systems in canonical forms (Ghalia, 1996; Boukezzoula et al., 2001; Allamehzadeh and Cheung, 2003), through control of fuzzy systems with parametric uncertainties (Gong et al., 2004; Chen et al., 1999; Chadli and El Hajjaji, 2005; Bai and Zhang, 2008) to delay-dependent fuzzy systems (Huang and Ho, 2007; Haibo et al., 2007; Chen and Liu, 2005). Applications include, among others, control of robotic manipulators (Hsu and Fu, 1994, 1995; Ham et al., 2000), magnetic bearing systems (Hong and Langari, 2000; Du et al., 2009), and vehicle lateral dynamics (El Hajjaji et al., 2006).

However, observer design and the contribution of the estimation error to stabilization using output feedback is rarely discussed. In particular, the performance of the observer designed for an approximate TS model and then applied to the original nonlinear system has not been studied for the case when the scheduling variables themselves have to be estimated. Therefore, in this paper, we investigate whether and when conclusions can be drawn when the design is based on an approximate fuzzy model and applied to the original nonlinear system that is approximated by the fuzzy model. To simplify the computations, a common quadratic Lyapunov function is used. Similar, although considerably more complex conditions can be derived if other Lyapunov functions or more relaxed conditions than those presented in this paper, are used.

The structure of the paper is as follows. Section 2 presents the models used and reviews classic results for the stability analysis of autonomous fuzzy systems. Section 3 investigates when the stability of a TS system implies the

⋆ This research is sponsored by Senter, Ministry of Economic Affairs of the Netherlands within the project Interactive Collaborative Information Systems (grant BSIK03024).

stability of the nonlinear system. These results serve as the basis for investigating the expected performance of the observer designed for the fuzzy model and applied to the nonlinear system in Section 4. We study the stabilization of the nonlinear system using a fuzzy state feedback controller in Section 5 and observer-based output feedback controller in Section 6. The different cases are illustrated using examples in the corresponding sections. Section 7 concludes the paper.

## 2. PRELIMINARIES

Consider the nonlinear system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u})\\ \boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{x},\boldsymbol{u})\end{aligned} \tag{1}$$

where $\boldsymbol{x}$ is the vector of the state variables, $\boldsymbol{u}$ is the input vector, $\boldsymbol{y}$ is the measurement vector. We assume that the variables are defined on a compact set $\mathcal{C}_{\boldsymbol{xuy}}$, i.e., $(\boldsymbol{x},\boldsymbol{u},\boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}$. A TS fuzzy approximation of this system can be obtained (e.g., by linearization) as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{f}^{\diamond}(\boldsymbol{x},\boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\\ \boldsymbol{y} &= \boldsymbol{h}^{\diamond}(\boldsymbol{x},\boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i)\end{aligned} \tag{2}$$

so that the approximation errors $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^{\diamond}$ and $\overline{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}^{\diamond}$ satisfy

$$\begin{aligned}\|\overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x},\,\boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}\\ \|\overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma_h + \delta_h\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x},\,\boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}\end{aligned} \tag{3}$$

where $\sigma_f$, $\sigma_h$, $\delta_f$, and $\delta_h$ are known nonnegative finite constants, and $\mathcal{C}_{\boldsymbol{xu}} = \{(\boldsymbol{x},\,\boldsymbol{u})|\exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x},\,\boldsymbol{u},\,\boldsymbol{y}) \in \mathcal{C}_{\boldsymbol{xuy}}\}$. In (2), $A_i$, $B_i$, $C_i$, $D_i$, $a_i$, and $d_i$, $i = 1, 2, \ldots, m$ represent the matrices and biases of the $i$th local linear model and $w_i$, $i = 1, 2, \ldots, m$ are the corresponding normalized membership functions, that depend on the scheduling variables $\boldsymbol{x}$, $\boldsymbol{u}$.

Throughout the paper it is assumed that the membership functions are normalized, i.e., $w_i(\boldsymbol{x},\boldsymbol{u}) \ge 0$, $\sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u}) = 1$, $\forall(\boldsymbol{x},\,\boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}}$. The symbols $I$ and 0, respectively, denote the identity and the zero matrices of the appropriate dimensions, $\mathcal{H}(A)$ represents the Hermitian of the matrix $A$, i.e., $\mathcal{H}(A) = A + A^T$, and $\|.\|$ denotes the Euclidean norm for vectors and the induced norm for matrices.

The nonlinear system (1) is now expressed as an uncertain TS system, given as:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\\ \boldsymbol{y} &= \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(C_i\boldsymbol{x} + D_i\boldsymbol{u} + d_i) + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\end{aligned} \tag{4}$$

where the uncertainties $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ satisfy (3).

Note that the approximation error on a compact set of variables always satisfies

$$\begin{aligned}\|\overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma'_f\\ \|\overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma'_h\end{aligned} \tag{5}$$

for some $\sigma'_f$ and $\sigma'_h$. However, as will be shown in the sequel, by using (3) whenever possible, less conservative conditions can be obtained.

*Remark 1:* In the robust fuzzy control literature (Bai and Zhang, 2008), for uncertain fuzzy systems in general the form

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{x})\Big[(A_i + \Delta A_i)\boldsymbol{x} + (B_i + \Delta B_i)\boldsymbol{u}\Big]$$

is used. However, the model (2) is more general, and therefore, (2) will be used in the sequel.

Our results are based on classic conditions (Wang et al., 1996; Tanaka et al., 1998) for the stability of autonomous fuzzy systems:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{z})A_i\boldsymbol{x} \tag{6}$$

where $A_i$, $i = 1, 2, \ldots, m$ represent the $i$th local linear model, $w_i$ is the corresponding normalized membership function, and $\boldsymbol{z}$ the vector of the scheduling variables, which may depend on the states, input, output, or other measured exogenous variables.

*Theorem 1. (Wang et al., 1996) System* (6) *is exponentially stable if there exists $P = P^T > 0$ so that*

$$\mathcal{H}(PA_i) < 0 \tag{7}$$

*for $i = 1, 2, \ldots, m$.*

Controller and observer design for fuzzy systems of the form (2) often leads to establishing the negative definiteness of double summations of the form

$$\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij}$$

with $\Upsilon_{ij}$, $i, j = 1, 2, \ldots m$ matrices of appropriate dimensions. In this paper we use the following relaxations for such sums (Wang et al., 1996):

*Theorem 2. Let $\Upsilon_{ij}$ be matrices of proper dimensions. Then,*

$$\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{z})w_j(\boldsymbol{z})\Upsilon_{ij} < 0 \tag{8}$$

*holds, if*

$$\begin{aligned}&\Upsilon_{ii} < 0 \qquad \text{for}\\ &\frac{1}{2}(\Upsilon_{ij} + \Upsilon_{ji}) < 0\end{aligned} \tag{9}$$

*for $i = 1, 2, \ldots, m$, $j = i+1, i+2, \ldots, m$.*

Note that similar, although more complex results can also be derived using other types of Lyapunov functions, as long as the conditions ensure the exponential stability of the TS system.

## 3. STABILITY ANALYSIS

Stability analysis of uncertain or perturbed nonlinear systems is in general investigated by using the Lyapunov function that establishes exponential stability of the nominal model for the uncertain system (Khalil, 2002). In this paper, we use a similar approach, i.e., the Lyapunov function that establishes stability of the fuzzy model is further used for the original nonlinear system.

For stability analysis, consider the *autonomous* nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{10}$$

that is approximated by the TS system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}^{\diamond}(\boldsymbol{x}) = \sum_{i=1}^{m} w_i(\boldsymbol{x}) A_i \boldsymbol{x} \tag{11}$$

such that each local matrix $A_i$, $i = 1, 2, \ldots, m$ is stable and the approximation error $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^{\diamond}$ satisfies

$$\|\overline{\boldsymbol{f}}(\boldsymbol{x})\| \leq \sigma_f + \delta_f \|\boldsymbol{x}\| \qquad \forall \boldsymbol{x} \tag{12}$$

where $\sigma_f$ and $\delta_f$ are nonnegative finite constants. Consider the Lyapunov function $V = \boldsymbol{x}^T P \boldsymbol{x}$. If there exist $P = P^T > 0$ and $Q = Q^T > 0$ so that the linear matrix inequality (LMI)

$$\mathcal{H}(PA_i) < -2Q, \quad i = 1, 2, \ldots, m \tag{13}$$

is satisfied, then, by applying the same Lyapunov function to the original nonlinear system (10), we obtain:

$$\begin{aligned}
\dot{V} &= \boldsymbol{x}^T \mathcal{H}\Big(P\Big(\sum_{i=1}^{m} w_i(\boldsymbol{x}) A_i \boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x})\Big)\Big) \\
&= \boldsymbol{x}^T \sum_{i=1}^{m} w_i(\boldsymbol{x}) \mathcal{H}(PA_i)\boldsymbol{x} + 2\boldsymbol{x}^T P \overline{\boldsymbol{f}}(\boldsymbol{x}) \\
&\leq -2\lambda_{\min}(Q)\|\boldsymbol{x}\|^2 + 2\lambda_{\max}(P)\delta_f\|\boldsymbol{x}\|^2 \\
&\quad + 2\lambda_{\max}(P)\sigma_f\|\boldsymbol{x}\| \\
&\leq -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)(1-\theta)\|\boldsymbol{x}\|^2 \\
&\quad - 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma_f)
\end{aligned}$$

with $\theta \in (0,1)$ arbitrarily chosen, and where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the eigenvalues with the smallest and largest absolute magnitude.

By analyzing the expression of $\dot{V}$, the following cases can be distinguished:

(1) $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f < 0)$ or $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f > 0)$: no conclusion can be drawn;

(2) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f = 0$: if the membership functions are sufficiently smooth, and $\boldsymbol{x} = 0$ is the only equilibrium point, based on LaSalle's invariance principle and Barbalat's lemma (see Theorem 4.4 and Lemma 8.2 of Khalil (2002)), $\boldsymbol{x} = 0$ is a globally asymptotically stable equilibrium point of the nonlinear system (10). This result is in general obtained when adaptive fuzzy controllers are designed. In stability analysis of TS systems, this case is rarely encountered.

(3) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f = 0$: the nonlinear system (10) has a globally exponentially stable equilibrium point in $\boldsymbol{x} = 0$. This result is found only if the approximation error is Lipschitz continuous in the states.

(4) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f > 0$ and $\sigma_f > 0$: the states of the nonlinear system (10) are uniformly ultimately bounded by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f} \frac{\sigma_f}{\theta} \tag{14}$$

or

$$\gamma < \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f} \sigma_f \tag{15}$$

This result is obtained as soon as the nonlinear system is approximated to a constant accuracy by the fuzzy model, i.e., the approximation error is bounded by a constant. Moreover, since in most cases only a constant upper bound on the approximation error can be determined, this is the most frequently found result.

The above cases are illustrated on the following example.

*Example 1.* Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} -1.1 & x_1^2 - x_2 \\ 2x_2 & -4.1 + x_2^2 \end{pmatrix} \boldsymbol{x} \tag{16}$$

with the state variables $x_1, x_2 \in [-1, 1]$.

The system has one equilibrium point, $\boldsymbol{x} = 0$. This equilibrium point is asymptotically stable on the domain defined by $x_1, x_2 \in [-1, 1]$. The stability is provable with the Lyapunov function $V = \boldsymbol{x}^T\boldsymbol{x}$. Note that if the sector nonlinearity approach is used to obtain an exact TS representation of this system, one of the local matrices is $\begin{pmatrix} -1.1 & 2 \\ 2 & -3.1 \end{pmatrix}$, which has a positive eigenvalue 0.1361. Therefore, the stability of the TS model so obtained cannot be established.

A TS approximation of the system (16) can be obtained using the approach of Kiriakidis (2007). Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1, x_2) | x_1, x_2 \in \{-1, 0, 1\}\}$. Therefore, 9 local models are obtained, and each one is asymptotically stable. Moreover, with this approximation we have the approximation error bounded by either $\|\overline{\boldsymbol{f}}\| \leq 0.58\|\boldsymbol{x}\|$, or $\|\overline{\boldsymbol{f}}\| \leq 0.53$ (computed numerically).

If the bound $\|\overline{\boldsymbol{f}}\| \leq 0.58\|\boldsymbol{x}\|$ is used, with $P$ and $Q$ computed[1] as $P = \begin{pmatrix} 14.4874 & 0.0211 \\ 0.0211 & 7.2243 \end{pmatrix}$, and $Q = \begin{pmatrix} 9.7100 & -2.9048 \\ -2.9048 & 10.4225 \end{pmatrix}$ the exponential stability of the nonlinear system is proven (case 3).

If the approximation error bound $\|\overline{\boldsymbol{f}}\| \leq 0.53$ is used, with $P = \begin{pmatrix} 0.4945 & 0.0379 \\ 0.0379 & 0.2188 \end{pmatrix}$, $Q = 0.3920\, I$ the ultimate bound $\gamma = 1.0469$ is obtained (case 4), i.e., the states converge within a ball with radius 1.04. □

## 4. STATE ESTIMATION

Besides their use in monitoring and control, the design of estimators in the presence of model uncertainties is one of the most important issues in fault detection and identification. However, observer design as such for nonlinear systems using TS fuzzy models when the TS model is only an approximation, and the guarantees that can be expected for the original nonlinear system are rarely discussed in the literature. It is important to note that in the context of robust output feedback fuzzy control, observers are used. However, it is then generally assumed that the

[1] To solve the LMI problems in this paper, the *SeDuMi* solver within the Yalmip toolbox (Löfberg, 2004) has been used.

controller compensates for or attenuates the estimation error resulting from the observer model–true system mismatch, without actually analyzing how this error affects the stability of the closed-loop system.

In this section, we consider that an observer is designed based on a TS approximation of a given nonlinear system. This observer is afterwards applied to the original nonlinear system. We investigate the guarantees that can be expected on the convergence of the estimation error, in particular, when the scheduling vector of the TS model depends on unmeasured states.

Therefore, consider the nonlinear system (1), with the approximation (4), so that the approximation errors $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^\diamond$ and $\overline{\boldsymbol{h}} = \boldsymbol{h} - \boldsymbol{h}^\diamond$ satisfy [2]

$$\begin{aligned} \|\overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma_f \qquad \forall \boldsymbol{x},\, \boldsymbol{u} \\ \|\overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\| &\le \sigma_h \qquad \forall \boldsymbol{x},\, \boldsymbol{u} \end{aligned} \tag{17}$$

where $\sigma_f$ and $\sigma_h$ are nonnegative finite constants. Recall that such a bound can always be obtained on a compact set, and therefore (17) is a valid assumption.

The observer considered in this section is of the form

$$\begin{aligned} \dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i) \end{aligned} \tag{18}$$

If the observer (18) is now used for the nonlinear system (1), the error dynamics can be expressed as:

$$\begin{aligned} \dot{\boldsymbol{e}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}^\diamond(\widehat{\boldsymbol{x}},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\boldsymbol{x},\boldsymbol{u})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\boldsymbol{e} - L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &\qquad + \sum_{i=1}^{m}(w_i(\boldsymbol{x},\boldsymbol{u}) - w_i(\widehat{\boldsymbol{x}},\boldsymbol{u}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})\Big(A_i\boldsymbol{e} - L_i \\ &\qquad \cdot \Big(\sum_{j=1}^{m} w_j(\boldsymbol{x},\boldsymbol{u})(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}) \\ &\qquad + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) - \sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(C_j\widehat{\boldsymbol{x}} + D_j\boldsymbol{u} + d_j)\Big)\Big) \\ &\qquad + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})\Big(A_i\boldsymbol{e} - L_i\Big(\sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})C_j\boldsymbol{e} \\ &\qquad + \sum_{j=1}^{m}(w_j(\boldsymbol{x},\boldsymbol{u}) - w_j(\widehat{\boldsymbol{x}},\boldsymbol{u}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) \\ &\qquad + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\Big)\Big) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \end{aligned}$$

or, simply as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_iC_j)\boldsymbol{e} \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \end{aligned} \tag{19}$$

with

$$\begin{aligned} \Delta_{wf} &= \sum_{i=1}^{m}(w_i(\boldsymbol{x},\boldsymbol{u}) - w_i(\widehat{\boldsymbol{x}},\boldsymbol{u}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\ \Delta_{wh} &= \sum_{j=1}^{m}(w_j(\boldsymbol{x},\boldsymbol{u}) - w_j(\widehat{\boldsymbol{x}},\boldsymbol{u}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) \end{aligned}$$

For the observer–TS fuzzy model mismatch, bounds similar to (3) are assumed:

$$\begin{aligned} \|\Delta_{wf}\| &\le \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\ \|\Delta_{wh}\| &\le \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| \end{aligned} \tag{20}$$

Using these bounds, in the worst case, we have

$$\begin{aligned} &\Big\| - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\Big\| \\ &\quad \le \max_i \|L_i\|(\sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \sigma_h) \\ &\qquad + \sigma_f + \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\ &\quad = \sigma + \delta\|\boldsymbol{e}\| \end{aligned} \tag{21}$$

with

$$\begin{aligned} \sigma &= \max_i \|L_i\|(\sigma_{wh} + \sigma_h) + \sigma_f + \sigma_{wf} \\ \delta &= \max_i \|L_i\|\delta_{wh} + \delta_{wf} \end{aligned} \tag{22}$$

To summarize, the error dynamics are:

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_iC_j)\boldsymbol{e} + \Delta \tag{23}$$

with $\|\Delta\| \le \sigma + \delta\|\boldsymbol{e}\|$, where $\delta$ and $\sigma$ are given by (22). Note however, that $\sigma$ depends on $L_i$, $i = 1, 2, \ldots, m$, the gains that have to be designed, and that in order to obtain the smallest possible bound on the estimation error, $\sigma$, and consequently $\|L_i\|$, $i = 1, 2, \ldots, m$ should be minimized.

Using the Lyapunov function $V = \boldsymbol{e}^TP\boldsymbol{e}$, similarly to Section 3, and assuming that there exist $P = P^T > 0$ and $Q = Q^T > 0$ so that

$$\begin{aligned} &\mathcal{H}(P(A_i - L_iC_i)) < -2Q \\ &\mathcal{H}(P(A_i - L_iC_j) + P(A_j - L_jC_i)) < -4Q \end{aligned} \tag{24}$$

for $i = 1, 2, \ldots, m$, $j = i + 1, i + 2, \ldots, m$, we get

[2] See Remark 2 for the explanation why a Lipschitz condition like (3) is not used.

$$\dot{V} = \boldsymbol{e}^T \mathcal{H}\Big(P \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u}) w_j(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i - L_i C_j)\boldsymbol{e}\Big)$$
$$+ 2\boldsymbol{e}^T P \Delta$$
$$\leq -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta)(1-\theta)\|\boldsymbol{x}\|^2$$
$$- 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma)$$

with $\theta \in (0,1)$. Based on the results presented in Section 3, by analyzing the expression of $\dot{V}$, when the observer (18) is applied to the nonlinear system (1), one of the following conclusions can be drawn regarding the estimation error:

(1) $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta < 0)$ or $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta = 0$ and $\sigma > 0)$: no conclusion can be drawn;
(2) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta = 0$ and $\sigma = 0$: under conditions similar to those in Section 3, the estimation error dynamics are asymptotically stable;
(3) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma = 0$: the nonlinear system (19) has a globally exponentially stable equilibrium point in $\boldsymbol{x} = 0$; However, this case can only be obtained if the fuzzy system is an exact representation of the nonlinear system, i.e., in (17) $\sigma_f, \sigma_h = 0$;
(4) $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma > 0$: the estimation error is uniformly ultimately bounded by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta}\frac{\sigma}{\theta} \qquad (25)$$

This is the result obtained in general.

The following example illustrates the computation of the bounds during observer design:

*Example 2.* Consider the nonlinear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_2^2 \end{pmatrix}\boldsymbol{x}$$
$$\boldsymbol{y} = [1\ 0]\boldsymbol{x} \qquad (26)$$

with $x_1, x_2 \in [-1, 1]$. This system is unstable.

A TS approximation of the system (26) is obtained using the approach of Kiriakidis (2007). Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1, x_2)|x_1, x_2 \in \{-1, 0, 1\}\}$, and 9 local models are obtained. The TS system can be written as:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m} w_i(\boldsymbol{x}) A_i \boldsymbol{x}$$
$$\boldsymbol{y} = [1\ 0]\boldsymbol{x} \qquad (27)$$

The approximation errors are $\|\overline{\boldsymbol{f}}\| \leq \sigma_f = 0.407$ and $\|\overline{\boldsymbol{h}}\| = \sigma_h = 0$. With these membership functions, we have $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - \boldsymbol{x}(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq 6.3$ and $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq 6.3\|\boldsymbol{e}\|$. Combining the two bounds, we can actually use $\|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))A_i\boldsymbol{x}\| \leq \alpha \cdot 6.3 + (1-\alpha)\cdot 6.3\|\boldsymbol{e}\|$, with $\alpha$ arbitrarily chosen in [0, 1]. Consequently, $\delta = (1-\alpha)\cdot 6.3$, and $\sigma = \alpha \cdot 6.3 + 0.407$.

Solving (37) such that simultaneously $\lambda_{\max}(P)$ is minimized and $\lambda_{\min}(P)$ and $\lambda_{\min}(Q)$ are maximized, one obtains: $\lambda_{\min}(P) = 0.33$, $\lambda_{\max}(P) = 0.33$ and $Q = I$. Consequently, $\delta = (1-\alpha)\cdot 6.3$, and $\sigma = \alpha \cdot 6.3 + 0.407$. With these values, the cases presented above become:

- for $\alpha < \frac{20}{21}$, i.e., $(1-\alpha)\cdot 6.3 > 0.3$ we have $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta < 0$ and no conclusion can be drawn
- since $\sigma_f > 0$, the conclusion of asymptotic stability (Cases 2) and 3) above) using the observer (18) is excluded.
- for $\alpha \geq \frac{20}{21}$, we obtain that the estimation error is uniformly ultimately bounded by

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta}\frac{\sigma}{\theta}$$
$$= \frac{0.33\cdot(6.3\alpha + 0.407)}{(1 - 2.08(1-\alpha))\theta}$$
$$< 2.2$$

with $\theta \in (0,\ 1)$ and $\alpha \in [\frac{20}{21},\ 1]$.

A large part of the value of the bound is due to observer–TS model mismatch, i.e., the dependency of the scheduling vector on the non-measured states. For instance, if the considered system is

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_1^2 \end{pmatrix}\boldsymbol{x}$$
$$\boldsymbol{y} = [1\ 0]\boldsymbol{x} \qquad (28)$$

instead of (26), the scheduling variable is $x_1$ only, which is measured, and consequently can be used in the observer. The difference with respect to system (26) is that the (2,2) element of the matrix depends on $x_1$ (that is measured) instead of $x_2$. Then, we have $\|\sum_{i=1}^{m}(w_i(\boldsymbol{z}) - w_i(\widehat{\boldsymbol{z}}))A_i\boldsymbol{x}\| = \|\sum_{i=1}^{m}(w_i(\boldsymbol{x}) - w_i(\boldsymbol{x}))A_i\boldsymbol{x}\| = 0$, $\Delta_{wf} = 0$, $\Delta_{wh} = 0$, and the bound on the estimation error is

$$\gamma = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}\frac{\sigma}{\theta} = \frac{0.13}{\theta} < 0.13$$

with $\theta \in (0,\ 1)$. □

Note that in Example 2 a common measurement matrix has been considered. If the measurement matrix is not common for all the rules, $\sigma$ depends on the $L_i$, $i = 1, 2, \ldots, m$ to be designed. In such a case, to facilitate the design, one can solve the multi-objective optimization problem:

$$\begin{aligned}
&\text{maximize } \lambda_{\min}(Q),\ \lambda_{\min}(P),\\
&\text{minimize } \lambda_{\max}(P),\ \|L_i\|, i = 1, 2, \ldots, m\\
&\text{subject to}\\
&P = P^T > 0\\
&Q = Q^T > 0\\
&\mathcal{H}(P(A_i - L_i C_i)) \leq -2Q\\
&\mathcal{H}(P(A_i - L_i C_j) + P(A_j - L_j C_i)) \leq -4Q
\end{aligned}$$

for $i = 1, 2, \ldots, m$, $j = i+1, i+2, \ldots, m$.

*Remark 2:* Recall that instead of using the bound (3), for the observer (18), the constant bound on the approximation error has been used. This is because, if the bounds on $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ for observer design are not constants, but linear in $\boldsymbol{x}$, then, with the observer (18), the state itself has to be treated as a disturbance that affects the error dynamics. This would lead to a much larger bound on the estimation error.

A possible approach to still attain asymptotic stability is when the approximation error functions $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ are known, and Lipschitz continuous in the states, i.e., there exist $\gamma_f, \gamma_h \geq 0$ such that $\|\overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) - \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}},\boldsymbol{u})\| \leq \gamma_f\|\boldsymbol{x} -$

$\widehat{\boldsymbol{x}}\|$, and $\|\overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}) - \overline{\boldsymbol{h}}(\widehat{\boldsymbol{x}},\boldsymbol{u})\| \leq \gamma_h \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|$. In this case, instead of the observer (18), the observer

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) + \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}},\boldsymbol{u}) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}},\boldsymbol{u})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i) + \overline{\boldsymbol{h}}(\widehat{\boldsymbol{x}},\boldsymbol{u})\end{aligned} \tag{29}$$

can be used.

Then, similarly to linear observer design for nonlinear systems with Lipschitz nonlinearities (Pertew et al., 2005, 2006), with $\gamma_f$ and $\gamma_h$ incorporated into $\delta$, asymptotic stability of the estimation error can be obtained.

*Example 3.* Consider the nonlinear system

$$\begin{aligned}\dot{\boldsymbol{x}} &= \begin{pmatrix} 0.33x_1^2 + x_1 & 1 \\ 1 & -1 \end{pmatrix}\boldsymbol{x} \\ y &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{30}$$

with $x_1,\, x_2 \in [-1,\, 1]$.

A TS approximation of the system (30) can be obtained as the two-rule fuzzy system:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{2} w_i(y)A_i\boldsymbol{x} \\ y &= [1\ 0]\boldsymbol{x}\end{aligned} \tag{31}$$

with $A_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $w_1(y) = \frac{1-y}{2}$, $w_2(y) = \frac{1+y}{2}$. The approximation error function is $\overline{\boldsymbol{f}} = \begin{pmatrix} 0.33x_1^2 & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{x}$. For this function, we have $\|\overline{\boldsymbol{f}}(\boldsymbol{x}) - \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}})\| \leq \|\boldsymbol{x} - \widehat{\boldsymbol{x}}\|$, i.e., $\delta = 1$, and $\sigma = 0$. Note that since the membership functions depend on a measured variable, there is no observer-model mismatch.

Solving (37), one obtains $L_1 = L_2 = \begin{pmatrix} 15.1 \\ 7.3 \end{pmatrix}$, $P = \begin{pmatrix} 0.2109 & -0.3183 \\ -0.3183 & 0.8475 \end{pmatrix}$, $Q = I$, $\lambda_{\min}(P) = 0.08$, and $\lambda_{\max}(P) = 0.97$. Then, using the observer

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{2} w_i(y)(A_i\widehat{\boldsymbol{x}} + L_i(y - \hat{y})) + \overline{\boldsymbol{f}}(\widehat{\boldsymbol{x}}) \\ \hat{y} &= [1\ 0]\widehat{\boldsymbol{x}}\end{aligned}$$

for the original nonlinear system (30), with a common quadratic Lyapunov function we obtain that there exist $P = P^T > 0$ and $Q = Q^T > 0$ such that $\lambda_{\min}(Q) - \lambda_{\max}(P)\delta > 0$ and $\sigma = 0$ (i.e., case 3), and the estimation error is asymptotically stable. □

## 5. STABILIZATION USING FULL STATE FEEDBACK

Development of sufficient conditions for the stabilization using full state feedback of uncertain TS fuzzy systems has received increasing interest in the last years (Gong et al., 2004; Du et al., 2009; Chen et al., 1999; Chadli and El Hajjaji, 2005), in particular for discrete-time systems. In this paper, we consider continuous-time TS systems. Instead of developing conditions to design controllers that asymptotically stabilize the system, we investigate what conclusions regarding the original nonlinear system can be drawn if a controller has already been designed for its fuzzy approximation. Therefore, consider the nonlinear system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) \tag{32}$$

that is approximated by the TS system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}^{\diamond}(\boldsymbol{x},\boldsymbol{u}) = \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \tag{33}$$

so that the approximation error $\overline{\boldsymbol{f}} = \boldsymbol{f} - \boldsymbol{f}^{\diamond}$ satisfies

$$\|\overline{\boldsymbol{f}}(\boldsymbol{x},\,\boldsymbol{u})\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\| \qquad \forall(\boldsymbol{x},\,\boldsymbol{u}) \in \mathcal{C}_{\boldsymbol{xu}} \tag{34}$$

with $\sigma_f$ and $\delta_f$ being nonnegative finite constants.

Although for observer design the state transition model may contain affine terms, for stabilization, the nonlinear system has to be approximated by a fuzzy model of the form (33), i.e., the local models may not be affine and the membership functions may not depend on the control input $\boldsymbol{u}$. This is firstly because stabilization to zero of affine fuzzy systems using a classical fuzzy state feedback can only be performed if the affine term is compensated for in each rule. Secondly, if the membership functions depend on the control input, when actually computing the input, an implicit equation has to be solved, which, depending on the membership functions, may be cumbersome.

Using a classical fuzzy state feedback control

$$\boldsymbol{u} = \sum_{i=1}^{m} w_i(\boldsymbol{x})K_i\boldsymbol{x}$$

we have the closed-loop fuzzy system:

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})(A_i + B_iK_j)\boldsymbol{x} \tag{35}$$

and the dynamics of the closed-loop nonlinear system can be described as

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})(A_i + B_iK_j)\boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \tag{36}$$

With a common quadratic Lyapunov function, the TS system (35) is globally exponentially stable, according to Theorem 2, if there exist $P = P^T > 0$ and $Q = Q^T > 0$ so that

$$\begin{aligned}&\mathcal{H}(P(A_i + B_iK_i)) < -2Q \\ &\mathcal{H}(P(A_i + B_iK_j) + P(A_j + B_jK_i)) < -4Q\end{aligned} \tag{37}$$

for $i = 1, 2, \ldots, m$, $j = i+1, i+2, \ldots, m$. With the same Lyapunov function applied to the original nonlinear system (1), we obtain:

$$\begin{aligned}\dot{V} &= \boldsymbol{x}^T\mathcal{H}\Big(P\Big(\sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\boldsymbol{x})G_{ij}\boldsymbol{x} + \overline{\boldsymbol{f}}(\boldsymbol{x})\Big)\Big) \\ &\leq -2(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)(1-\theta)\|\boldsymbol{x}\|^2 \\ &\quad - 2\|\boldsymbol{x}\|(\theta(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f)\|\boldsymbol{x}\| - \lambda_{\max}(P)\sigma_f)\end{aligned}$$

with $\theta \in (0,1)$ arbitrarily chosen and $G_{ij} = A_i + B_iK_j$.

By analyzing the expression of $\dot{V}$, the following cases can be distinguished:

(1) $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f < 0)$ or $(\lambda_{\min}(Q) - \lambda_{\max}(P)\delta_f = 0$ and $\sigma_f > 0)$: no conclusion can be drawn;

(2) $\lambda_{\min}(Q)-\lambda_{\max}(P)\delta_f=0$ and $\sigma_f=0$: if the membership functions are sufficiently smooth, and $\boldsymbol{x}=0$ is the only equilibrium point, under conditions similar to those in Section 3, $\boldsymbol{x}=0$ is a globally asymptotically stable equilibrium point of the nonlinear system (36). Such results are in general obtained when adaptive fuzzy controllers are designed.
(3) $\lambda_{\min}(Q)-\lambda_{\max}(P)\delta_f>0$ and $\sigma_f=0$: the nonlinear system (36) has a globally exponentially stable equilibrium point in $\boldsymbol{x}=0$. Note that this result can only be obtained if the approximation error is Lipschitz continuous in the states.
(4) $\lambda_{\min}(Q)-\lambda_{\max}(P)\delta_f>0$ and $\sigma_f>0$: the states of the nonlinear system (36) are uniformly ultimately bounded by
$$\gamma=\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)-\lambda_{\max}(P)\delta_f}\frac{\sigma_f}{\theta}. \tag{38}$$
with $\theta\in(0,\,1)$.

It is important to note that in robust fuzzy control the affine term in (34) is in general considered to be an external disturbance affecting the system, and not a model mismatch, i.e, the uncertainty is presumed to affect only the matrices $A_i$ and $B_i$, $i=1,2,\ldots,m$. Nevertheless, even if the disturbance is due to model mismatch, robust controllers that are able to attenuate its effect can be designed.

If the controller has already been designed using (37), only the above conditions can be verified. However, if the controller is to be designed, then, in order to obtain a bound as small as possible one can also solve the multiobjective optimization problem:

$$\begin{aligned}
&\text{maximize } \lambda_{\min}(Q),\ \lambda_{\min}(P),\\
&\text{minimize } \lambda_{\max}(P),\\
&\text{subject to}\\
&P=P^T>0\\
&Q=Q^T>0\\
&\mathcal{H}(P(A_i+B_iK_i))<-2Q,\quad i=1,2,\ldots,m\\
&\mathcal{H}(P(A_i+B_iK_j)+P(A_j+B_jK_i))\le-4Q
\end{aligned} \tag{39}$$
for $i=1,2,\ldots,m$, $j=i+1,i+2,\ldots,m$.

The controller design is illustrated using the following example.

*Example 4.* Consider the nonlinear system
$$\dot{\boldsymbol{x}}=\begin{pmatrix}1.1 & x_1^2+0.1\\ -x_1-1 & -3-x_2^2\end{pmatrix}\boldsymbol{x}+\begin{pmatrix}1\\0\end{pmatrix}\boldsymbol{u} \tag{40}$$
with $x_1,\,x_2\in[-1,\,1]$. The autonomous system is unstable.

A TS approximation of the system (40) is obtained using the approach of Kiriakidis (2007). Normalized triangular membership functions are chosen, that attain their maximum in the points defined by $\{(x_1,x_2)|x_1,x_2\in\{-1,0,1\}\}$. The TS system can be written as:
$$\dot{\boldsymbol{x}}=\sum_{i=1}^{9}w_i(\boldsymbol{x})(A_i\boldsymbol{x}+Bu) \tag{41}$$
The approximation errors can be written as $\|\overline{\boldsymbol{f}}\|\le\sigma_f+\delta_f\|\boldsymbol{x}\|=0.407\alpha+0.48(1-\alpha)\|\boldsymbol{x}\|$, with $\alpha$ arbitrarily chosen in [0, 1], and $\|\overline{\boldsymbol{h}}\|=\sigma_h+\delta_h\|\boldsymbol{x}\|=0$.

By simply solving the LMI feasibility problem

*Find $P=P^T>0$, $Q=Q^T>0$, $K_i$, $i=1,2,\ldots,m$ so that* (37) *is satisfied*

one obtains $P=\begin{pmatrix}6.5 & 0.33\\ 0.33 & 0.38\end{pmatrix}$, $\lambda_{\min}(P)=0.36$, $\lambda_{\max}(P)=6.52$, $Q=I$. With these results, the following cases can be distinguished:

(1) if $\alpha$ is chosen such that $\alpha<0.69$, then we have $\lambda_{\min}(Q)-\lambda_{\max}(P)\delta_f>0$ and therefore no conclusion can be drawn
(2) for $\alpha>0.69$, we have Case 4, i.e., the states of the controlled nonlinear system (40), using the controller designed for the fuzzy system (41) are ultimately uniformly bounded by
$$\begin{aligned}\gamma&=\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)-\lambda_{\max}(P)\delta}\frac{\sigma}{\theta}\\ &=\frac{11.3\alpha}{(1-3.13(1-\alpha))\theta}<11.3\end{aligned}$$
with $\alpha\in[0.69,\,1]$ and $\theta\in(0,\,1)$.

Solving (39), i.e., minimizing[3] $\lambda_{\max}(P)$ and maximizing $\lambda_{\min}(Q)$ and $\lambda_{\min}(P)$, one obtains: $P=\begin{pmatrix}0.20 & 0.002\\ 0.002 & 0.17\end{pmatrix}$, $\lambda_{\min}(P)=0.17$, $\lambda_{\max}(P)=0.20$, and $Q=I$.

With these values, depending on the choice of $\alpha$, we have the following cases:

(1) for $\alpha=0$ we have $\sigma_f=0$ and $\lambda_{\min}(Q)-\lambda_{\max}(P)\delta>0$ and therefore the states of the nonlinear system converge exponentially to 0
(2) for $\alpha=1$, i.e., when a constant approximation error is considered, the states of the nonlinear system (40) are uniformly ultimately bounded by $\gamma=\frac{0.088}{\theta}$, with $\theta\in(0,\,1)$, i.e., $\gamma<0.088$.
(3) otherwise, we obtain that the states are uniformly ultimately bounded by
$$\begin{aligned}\gamma&=\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)-\lambda_{\max}(P)\delta}\frac{\sigma}{\theta}\\ &=\frac{0.088\alpha}{(1-0.48(1-\alpha))\theta}\end{aligned}$$
with $\theta\in(0,\,1)$ and $\alpha\in(0,\,1)$.

As illustrated above, by solving the optimization problem together with the design problem, not only a lower bound, but even exponential convergence of the nonlinear system can be obtained. □

## 6. STABILIZATION USING OBSERVER-BASED CONTROL

Although output feedback control is often considered in robust fuzzy control, it is in general assumed that the controller is able to compensate for or attenuate the disturbance resulting from the mismatch between the model used by the observer and the true system, without

[3] To solve this problem, a single objective function $\lambda_{\max}(P)-\lambda_{\min}(P)-\lambda_{\min}(Q)$ has been minimized.

explicitly analyzing this model mismatch. In this section, although we do not design robust controllers, we analyze the disturbance due to the mismatch and investigate what guarantees can be given.

Note that also in this case, the membership function cannot depend on the control input, and the state transition function cannot have an affine term, i.e., the same restrictions as in Section 5 apply. Therefore, the approximation considered is (2), with the approximation errors bounded as (3).

The observer is of the form

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(C_i\widehat{\boldsymbol{x}} + D_i\boldsymbol{u} + d_i)\end{aligned} \tag{42}$$

and the controller used is

$$\boldsymbol{u} = \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})K_i\widehat{\boldsymbol{x}} \tag{43}$$

The estimation error for the nonlinear system can be derived as:

$$\begin{aligned}\dot{\boldsymbol{e}} &= \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u}) - \boldsymbol{f}^{\diamond}(\widehat{\boldsymbol{x}},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})(A_i\boldsymbol{e} - L_i(\boldsymbol{y} - \widehat{\boldsymbol{y}})) \\ &\quad + \sum_{i=1}^{m} (w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}) + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})\Big(A_i\boldsymbol{e} - L_i \\ &\quad \cdot \Big(\sum_{j=1}^{m} w_j(\boldsymbol{x})(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u}) \\ &\quad - \sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}})(C_j\widehat{\boldsymbol{x}} + D_j\boldsymbol{u} + d_j)\Big)\Big) \\ &\quad + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})\Big(A_i\boldsymbol{e} - L_i\Big(\sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}})C_j\boldsymbol{e} \\ &\quad + \sum_{j=1}^{m} (w_j(\boldsymbol{x}) - w_j(\widehat{\boldsymbol{x}}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j) \\ &\quad + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})\Big)\Big) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\end{aligned}$$

so

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}})w_j(\widehat{\boldsymbol{x}})(A_i - L_iC_j)\boldsymbol{e} \\ &\quad - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\end{aligned} \tag{44}$$

with

$$\begin{aligned}\Delta_{wf} &= \sum_{i=1}^{m} (w_i(\boldsymbol{x}) - w_i(\widehat{\boldsymbol{x}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u}) \\ \Delta_{wh} &= \sum_{j=1}^{m} (w_j(\boldsymbol{x}) - w_j(\widehat{\boldsymbol{x}}))(C_j\boldsymbol{x} + D_j\boldsymbol{u} + d_j)\end{aligned}$$

Since the goal is also to stabilize the system, in this case the bounds on $\overline{\boldsymbol{f}}$ and $\overline{\boldsymbol{h}}$ can contain a term that is Lipschitz in $\boldsymbol{x}$. Moreover, one could also use bounds on $\Delta_{wf}$ and $\Delta_{wh}$ such as

$$\begin{aligned}\|\Delta_{wf}\| &\le \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| + \eta_{wf}\|\boldsymbol{x}\| \\ \|\Delta_{wh}\| &\le \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \eta_{wh}\|\boldsymbol{x}\|\end{aligned} \tag{45}$$

For the simplicity of the computations, the following bounds are assumed:

$$\begin{aligned}\|\Delta_{wf}\| &\le \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\ \|\Delta_{wh}\| &\le \sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\|\end{aligned} \tag{46}$$

Then, in the worst case, the bound on the observer-system mismatch can be derived as:

$$\begin{aligned}&\Big\| - \sum_{i=1}^{m} w_i(\widehat{\boldsymbol{x}})L_i(\Delta_{wh} + \overline{\boldsymbol{h}}(\boldsymbol{x},\boldsymbol{u})) + \Delta_{wf} + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\Big\| \\ &\quad \le \max_i \|L_i\|(\sigma_{wh} + \delta_{wh}\|\boldsymbol{e}\| + \sigma_h + \delta_h\|\boldsymbol{x}\|) \\ &\qquad + \sigma_f + \delta_f\|\boldsymbol{x}\| + \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| \\ &\quad \le \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|\end{aligned} \tag{47}$$

with

$$\begin{aligned}\sigma_e &= \max_i \|L_i\|(\sigma_{wh} + \sigma_h) + \sigma_f + \sigma_{wf} \\ \delta_e &= \max_i \|L_i\|\delta_{wh} + \delta_{wf} \\ \eta_e &= \max_i \|L_i\|\delta_h + \delta_f\end{aligned} \tag{48}$$

In fact:

$$\dot{\boldsymbol{e}} = \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\widehat{\boldsymbol{x}})w_j(\widehat{\boldsymbol{x}})(A_i - L_iC_j)\boldsymbol{e} + \Delta_e \tag{49}$$

with $\|\Delta_e\| \le \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|$.

Second, the closed-loop dynamics using the estimate-based control law is:

$$\begin{aligned}\dot{\boldsymbol{x}} &= \sum_{i=1}^{m} w_i(\boldsymbol{x})(A_i\boldsymbol{x} + B_i\sum_{j=1}^{m} w_j(\widehat{\boldsymbol{x}})K_j\widehat{\boldsymbol{x}}) \\ &\quad + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u}) \\ &= \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(\boldsymbol{x})w_j(\widehat{\boldsymbol{x}})((A_i + B_iK_j)\boldsymbol{x} + B_iK_j\boldsymbol{e}) \\ &\quad + \overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\end{aligned} \tag{50}$$

with

$$\|\overline{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{u})\| \le \sigma_f + \delta_f\|\boldsymbol{x}\| \tag{51}$$

Combining the dynamics of the estimation error and the state, we get

$$\begin{aligned}\begin{pmatrix}\dot{\boldsymbol{e}} \\ \dot{\boldsymbol{x}}\end{pmatrix} &= \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} w_i(\widehat{\boldsymbol{x}})w_j(\boldsymbol{x})w_k(\widehat{\boldsymbol{x}}) \\ &\quad \cdot \begin{pmatrix} A_i - L_iC_k & 0 \\ K_k & A_j + B_jK_k\end{pmatrix}\begin{pmatrix}\boldsymbol{e} \\ \boldsymbol{x}\end{pmatrix} + \Delta\end{aligned} \tag{52}$$

with

$$\Delta = \begin{pmatrix} \Delta_e \\ \overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u}) \end{pmatrix} \tag{53}$$

Knowing that $\|\Delta_e\| \leq \sigma_e + \delta_e\|\boldsymbol{e}\| + \eta_e\|\boldsymbol{x}\|$ and $\|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\|$, we have

$$\begin{aligned} \|\Delta\| \leq& \|\Delta_e\| + \|\overline{\boldsymbol{f}}(\boldsymbol{x}, \boldsymbol{u})\| \\ \leq& \sigma_e + \sigma_f + \delta_e\|\boldsymbol{e}\| + (\eta_e + \delta_f)\|\boldsymbol{x}\| \\ \leq& \sigma + \delta \left\| \begin{matrix} \boldsymbol{e} \\ \boldsymbol{x} \end{matrix} \right\| \end{aligned}$$

where $\sigma = \sigma_e + \sigma_f$ and $\delta = \sqrt{2}\max\{\delta_e, \eta_e + \delta_f\}$.

For the above bounds, the same cases can be distinguished as in the previous section. However, it has to be noted that firstly, Case 2) and Case 3) (see Section 3) in practice will only be obtained if the fuzzy model is an exact representation of the nonlinear system and the membership functions do not depend on unmeasured variables. Secondly, the bound obtained in Case 4) is very conservative, and therefore in practical cases the applied output feedback obtains better results than those that can be concluded based on this bound. Moreover, also due to the conservativeness of the result, the design of the output feedback control such that some desired bounds are satisfied is not practical. However, the bounds can also be computed after designing the observer and controller, and therefore be used to establish guarantees for the closed-loop system.

The following example illustrates the computation of the bounds for output feedback control:

*Example 5.* Consider the nonlinear system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_2^2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{u} \\ \boldsymbol{y} &= [1\ 0]\boldsymbol{x} \end{aligned} \tag{54}$$

with $x_1, x_2 \in [-1, 1]$.

A TS approximation of this system is obtained as in Example 2, where the approximation errors are $\|\overline{\boldsymbol{f}}\| \leq \sigma_f + \delta_f\|\boldsymbol{x}\| = 0.407\alpha + 0.48(1-\alpha)\|\boldsymbol{x}\|$, $\alpha \in [0,\ 1]$ and $\|\overline{\boldsymbol{h}}\| = 0$. With the same membership functions as in Example 2, we also have $\Delta_{wf} \leq \sigma_{wf} + \delta_{wf}\|\boldsymbol{e}\| = \beta \cdot 6.3 + (1-\beta)\cdot 6.3\|\boldsymbol{e}\|$, with $\beta \in [0,\ 1]$. Since the measurement matrix is common for all the rules, the equations can be simplified, and we have $\|\Delta_{wh}\| = 0$. Consequently, $\sigma_e = 0.407\alpha + 6.3\beta$, $\delta_e = 6.3(1-\beta)$, and $\eta_e = 0.48(1-\alpha)$, and $\sigma = 0.814\alpha + 6.3\beta$, and $\delta = \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\}$.

Solving the problem

*Find* $P = P^T > 0$, $Q = Q^T > 0$ *such that*

$$\mathcal{H}\Big(P \begin{pmatrix} A_i - L_i C_k & 0 \\ K_k & A_j + B_j K_k \end{pmatrix}\Big) < -2Q$$

*for* $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, m$

we obtain $P = \begin{pmatrix} 12.53 & 0 & 0 & 0 \\ 0 & 12.53 & 0 & 0 \\ 0 & 0 & 9.06 & 2.90 \\ 0 & 0 & 2.90 & 1.03 \end{pmatrix}$ and $Q = I$, $\lambda_{\min}(P) = 0.09$, and $\lambda_{\max}(P) = 12.53$. With these values, we have the bound on the state and estimation error

$$\gamma = \frac{147.8(0.814\alpha + 6.3\beta)}{\theta(1 - \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\}} \tag{55}$$

under the condition that

$$1 - \sqrt{2}\max\{6.3(1-\beta), 0.96(1-\alpha)\} > 0$$

and $\alpha, \beta, \theta \in (0,\ 1)$. It can be easily seen that this bound is very large, irrespective of the values chosen for $\alpha$, $\beta$, such that (55) is satisfied. However, a large part of this bound is due to the observer-model error. For instance, consider the system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 1.1 & x_1^2 + 0.1 \\ -x_1 - 1 & -3 + x_1^2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{u} \\ \boldsymbol{y} &= [1\ 0]\boldsymbol{x} \end{aligned} \tag{56}$$

with $x_1, x_2 \in [-1, 1]$. The difference with respect to the system (54) is that the $(2,2)$ element of the matrix depends on $x_1$, instead of $x_2$. For this system, the membership functions will only depend on $x_1 = y$, i.e., on the measured variable. Therefore, in the membership functions of the observer, we can use its true value, and consequently $\Delta_{wf} = \Delta_{wh} = 0$. Moreover, a solution such that $\lambda_{\max}(P) = 1.05$, $\lambda_{\min}(P) = 0.73$, and $\lambda_{\min}(Q) = 1$ can also be obtained. With these values, the bound on the estimation error becomes

$$\begin{aligned} \gamma &= \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q) - \lambda_{\max}(P)\delta} \frac{\sigma}{\theta} \\ &= \frac{0.512\alpha}{(1 - 0.5(1-\alpha))\theta} \end{aligned}$$

with $\theta \in (0, 1)$ and $\alpha \in [0, 1]$. It can easily be seen that for $\alpha = 0$, this bound is actually 0, and therefore both the states of the nonlinear system and the estimation error converge to 0. □

## 7. CONCLUSIONS

In this paper we have investigated whether and when stability guarantees can be obtained when an observer and a controller are designed for a fuzzy approximation of a nonlinear system and applied to the original nonlinear system. If the nonlinear system can be exactly represented or approximated up to a term that is Lipschitz continuous, under certain conditions, the dynamics of closed-loop system are globally asymptotically stable. Otherwise, the variables of the system are in general bounded. An upper bound can be computed based on the approximation error and the Lyapunov function used. We have also studied how the guarantees depend on the approximation error and on the mismatch between the observer–TS model and the true system. In our future research we will investigate whether the results can be improved.

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