Unknown input estimation for nonlinear descriptor systems via LMIs and Takagi-Sugeno models*

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***Abstract*— This paper presents an unknown inputs observer for nonlinear descriptor systems. The approach uses the Takagi-Sugeno representation of the nonlinear model. In order to obtain strict linear matrix inequalities a novel observer structure is given. Thus the conditions can be efficiently solved via convex optimization techniques. A numerical example is provided to illustrate the performance of the proposed approach.**

I. Introduction

Systems are subject to known inputs (designed control input) and unknown inputs (disturbances, measurement noise, modeling uncertainties, etc.). Designing observers for both the states of the system and unknown inputs (UIs) is an important task in robust control, monitoring and fault-tolerant control [1]–[5]. Estimating UIs also reduces the number of sensors to be used. For instance, in biomechanics, the estimation of an UIs such as the joint torques and angular velocities avoids the use of sensors on the person under study [6].

A proportional-integral (PI) observer for the estimation of both the state and the UIs has been developed in [7], where it is assumed that $\dot{d}=0$ (d is the unknown input); later a proportional multi-integral (PMI) observer has been proposed in [4], [8]. This observer allows estimating polynomial inputs. There exist several results on the estimation of UI for standard linear systems or linear descriptor systems [2], [4], [9]–[11]. Generally, the conditions are given in terms of linear matrix inequality (LMI) conditions together with equality constraints.

On the other hand, the number of results on the analysis and synthesis of nonlinear models via Takagi-Sugeno (TS) models have increased during the last twenty years. A TS model is a convex combination of linear models using nonlinear membership functions (MFs) [12]. The Lyapunov's direct method is employed for analysis and design. In general, the conditions can be formulated as LMIs, which can be efficiently solved via convex optimization techniques [13], [14]. The main results in this field have been collected in [15], [16]. In addition, using the sector nonlinearity approach guarantees that the obtained TS model is an exact representation of the nonlinear one in a compact set of the state space [17]. Nonetheless, when employing the sector nonlinearity methodology, the p non-linear terms in the nonlinear model generate 2^p local models (vertices), i.e., for a large number of non-constant terms the number of vertices rapidly increases.

The TS descriptor model has been introduced n [18] for those systems whose equations naturally appear as nonlinear descriptor models [19]. A TS descriptor model may help to alleviate the high number of vertices in the TS representation [20]. In addition, the final TS representation keeps the original structure of the given nonlinear model [21]–[23].

Under the TS-LMI framework, UI observers have been presented in [6], [24]–[27]. An observer that estimates the state and minimizes the influence of the UIs has been proposed in [28]. An application of PI observers for TS descriptor models has been addressed [6], under the assumption $\dot{d}\approx 0$; however the conditions are in terms of bilinear matrix inequalities (BMIs). In [24], conditions for a PMI observer of standard TS models have been given, where polynomial UIs can be estimated at the cost of enlarging the LMI problem to be solved.

This paper provides a new observer structure for the estimation of UIs; this novel structure is achieved by introducing an auxiliary variable in the estimated state vector [29]. This procedure leads to strict LMI conditions, thus overcoming the results in [6]. In addition, a large family of inputs is taken into account.

The rest of the paper is organized as follows: Section II provides the notation used throughout the work, states the problem under study and motivates this research; Section III presents the design of an unknown input observer via strict LMI conditions; Section IV illustrates the proposed approach via a numerical example; Section V concludes the paper.

II. Notation and problem statement

A. Notation

When dealing with convex sums of matrices Υ_i, Υ_{ij}, $i,j\in\{1,2,\ldots,r\}$ the following shorthand notation will be employed: $\Upsilon_h=\sum_{i=1}^{r}h_i(\bullet)\Upsilon_i$, $\Upsilon_h^{-1}=\left(\sum_{i=1}^{r}h_i(\bullet)\Upsilon_i\right)^{-1}$, and $\Upsilon_{hh}=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\bullet)h_j(\bullet)\Upsilon_{ij}$. Subscripts may change

*Research supported by the Ministry of Higher Education and Research, the CNRS, the Nord-Pas-de-Calais Region (grant No. 2013_11067), and a grant of the Technical University of Cluj-Napoca. The authors gratefully acknowledge the support of these institutions.

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according to the associated MF. An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side:

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix} = \begin{bmatrix} A & (*) \\ B & C \end{bmatrix}, \quad A+B+A^T+B^T = A+B+(*).$$

The following relaxation lemma is employed to drop off the MFs from expressions in order to obtain an LMI formulation.

Relaxation Lemma [30]: Let Υ_{ij}^k, $i,j \in \{1,2,\ldots,r\}$, $k \in \{1,2,\ldots,r_e\}$ be matrices of appropriate dimensions. Then $\Upsilon_{hh}^v = \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^{r_e} h_i(z) h_j(z) v_k(z) \Upsilon_{ij}^k < 0$ holds if

$$\begin{aligned} &\Upsilon_{ii}^k < 0, \quad \forall i,k, \\ &\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ij}^k < 0, \quad i \neq j, \quad \forall k. \end{aligned} \tag{1}$$

B. *Problem statement*

Consider the following continuous-time nonlinear model in the descriptor form [19]:

$$\begin{aligned} E(x)\dot{x}(t) &= A(x)x(t) + B(x)u(t) + M(x)d(t) \\ y(t) &= C(x)x(t) + G(x)d(t), \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ stands for the control input, $d \in \mathbb{R}^q$ is the unknown input vector, $y \in \mathbb{R}^o$ is the output of the system. Matrices $A(x)$, $B(x)$, $C(x)$, $M(x)$, $G(x)$, and $E(x)$ are assumed to be smooth in a compact set Ω_x of the state space including the origin. Moreover, in this work the matrix $E(x)$ is assumed to be regular in Ω_x; this is motivated by mechanical systems where $E(x)$ the inertia matrix, and therefore it is nonsingular [31]. Arguments will be omitted when their meaning is clear.

In this work, we use the sector nonlinearity methodology [17], since it allows obtaining a convex representation of (2) that exactly represents the original nonlinear model. Therefore, the p non-constant terms in right-hand side matrices $(A(x), B(x), C(x), M(x), G(x))$ are captured by the membership functions (MFs) $h_i(z)$, $i \in \{1,2,\ldots,2^p\}$. The same applies for p_e nonlinear terms in $E(x)$: $v_k(z)$, $k \in \{1,2,\ldots,2^{p_e}\}$. The MFs hold the convex sum property, i.e., $\sum_{i=1}^r h_i(z) = 1$, $h_i(z) \geq 0$, $\sum_{k=1}^{r_e} v_k(z) = 1$, and $v_k(z) \geq 0$, with $r = 2^p$ and $r_e = 2^{p_e}$. The vector $z(t) \in \mathbb{R}^{p+p_e}$ is called the premise vector and it is assumed to be known.

Thus via the sector nonlinearity approach, the nonlinear model (2) gives the following Takagi-Sugeno descriptor form [21]:

$$\begin{aligned} \sum_{k=1}^{r_e} v_k(z) E_k \dot{x} &= \sum_{i=1}^r h_i(z)\left(A_i x + B_i u + M_i d\right) \\ y &= \sum_{i=1}^r h_i(z)\left(C_i x + G_i d\right), \end{aligned} \tag{3}$$

where matrices $(A_i, B_i, M_i, C_i, G_i)$ and E_k represent the *i*-th linear right-side model and the *k*-th linear left-side model of the TS descriptor model. The linear models are blended together via the MFs $h_i(z)$, $i \in \{1,2,\ldots,r\}$, $v_k(z)$, $k \in \{1,2,\ldots,r_e\}$ [15]. Using the shorthand notation, (3) is expressed as:

$$E_v \dot{x} = A_h x + B_h u + M_h d, \quad y = C_h x + G_h d.$$

The goal of this paper is to design an observer capable of estimating both the state $x(t)$ and the UI $d(t)$. To that end, consider that the UIs are given by an exo-system, with the dynamics $\dot{d} = Sd$, where $S \in \mathbb{R}^{q \times q}$ is a known matrix. Using an extended state vector $x^e = \begin{bmatrix} x^T & d^T \end{bmatrix}^T \in \mathbb{R}^{n+q}$ the TS descriptor model (3) is rewritten as follows:

$$\begin{aligned} E_v^e \dot{x}^e &= A_h^e x^e + B_h^e u \\ y &= C_h^e x^e, \end{aligned} \tag{4}$$

with $E_v^e = \begin{bmatrix} E_v & 0 \\ 0 & I_q \end{bmatrix} \in \mathbb{R}^{(n+q)\times(n+q)}$, $B_h^e = \begin{bmatrix} B_h \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+q)\times m}$, $A_h^e = \begin{bmatrix} A_h & M_h \\ 0 & S \end{bmatrix} \in \mathbb{R}^{(n+q)\times(n+q)}$, and $C_h^e = \begin{bmatrix} C_h & G_h \end{bmatrix} \in \mathbb{R}^{o\times(n+q)}$.

C. *Motivation*

In this paper the descriptor matrix $E(x) = E_v = \sum_{k=1}^{r_e} v_k(z) E_k$ is assumed to be regular in Ω_x. Of course, classical tools could be used in the sense that (3) can be written as:

$$\begin{aligned} \dot{x} &= (E_v)^{-1} A_h x + (E_v)^{-1} B_h u + (E_v)^{-1} M_h d \\ &= \tilde{A}(x)x + \tilde{B}(x)u + \tilde{M}(x)d, \end{aligned} \tag{5}$$

where $\tilde{A}(x) = E_v^{-1}A_h$, $\tilde{B}(x) = E_v^{-1}B_h$, and $\tilde{M}(x) = E_v^{-1}M_h$. The same reasoning applies for the extended system (4). Nevertheless, coming back to a standard structure (5) is generally accompanied by an increase of nonlinear terms and therefore with possibly unfeasible or computationally intractable LMI constraints [15], [16].

Example 1. Consider a nonlinear descriptor model (2), with the following matrices:

$$E(x)=\begin{bmatrix}0.87 & 0.33+0.5(1-2\eta)\\ 0.53-\delta(1-2\eta) & 0.95\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},$$
$$A(x)=\begin{bmatrix}-0.81 & 0.83+\delta\cos(x_1)\\ -0.74 & 0.57\end{bmatrix},\ M(x)=\begin{bmatrix}2 & 1+\delta\cos(x_1)\\ 1 & -0.5\end{bmatrix},$$
$$C(x)=\begin{bmatrix}1.5+0.5\cos(x_1) & 0\\ 0 & 0.1\end{bmatrix},\ G(x)=\begin{bmatrix}1 & -0.5\\ 0.2+0.5\cos(x_1) & -0.4\end{bmatrix},$$

with $\eta=1/(1+x_1^2)$, δ is a real-valued parameter known a priori. Using the sector nonlinearity approach, a TS descriptor model can be constructed with $r=2$ due to the term $\cos(x_1)$; $r_e=2$ due to $\eta=1/(1+x_1^2)$. In total 4 vertices are needed to exactly represent the original nonlinear system. On the other hand, obtaining a standard state space requires $E^{-1}(x)$, which is possible since $E(x)$ is not singular. Then, (5) is computed with

$$\tilde{A}(x)=z_2\begin{bmatrix}0.95 & -0.33-0.5z_1\\ -0.53+\delta z_1 & 0.87\end{bmatrix}A(x)$$
$$\tilde{B}(x)=z_2\begin{bmatrix}0.95 & -0.33-0.5z_1\\ -0.53+\delta z_1 & 0.87\end{bmatrix}B$$
$$\tilde{M}(x)=z_2\begin{bmatrix}0.95 & -0.33-0.5z_1\\ -0.53+\delta z_1 & 0.87\end{bmatrix}M(x),$$

where $z_2=\left(0.6516+0.33\delta z_1-0.2650z_1+0.5\delta z_1^2\right)^{-1}$ and $z_1=(-1+x_1^2)/(1+x_1^2)$. Note that all the nonlinear terms are in the right-hand side of (5). Since there are three non-constant terms — z_1 and z_2 from $E^{-1}(x)$ and $z_3=\cos(x_1)$ from $(A(x),M(x),C(x),G(x))$ —, using the sector nonlinearity approach $r=2^3=8$ vertices are obtained. Moreover for this example the input matrix moves from constant in (3) to state-dependent in (5). Both facts increase the number of LMI conditions to be verified and generally reduce the set of possible solutions. This example is continued in section IV. ♦

Several works [6], [21], [23] have shown that keeping the structure of the observer close to the nonlinear model can significantly improve the quality of the results.

In order to work with the initial structure (3), a classical approach is to use an extended descriptor structure embedding equation (4) as a differential algebraic equation (DAE). Then, define $\bar{x}=\begin{bmatrix}x^e\\ \dot{x}^e\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}$ and the model writes directly:

$$\bar{E}\dot{\bar{x}}=\bar{A}_{hv}\bar{x}+\bar{B}_h u,\qquad y=\bar{C}_h\bar{x},\tag{6}$$

with, $\bar{E}=\begin{bmatrix}I_{n+q} & 0\\ 0 & 0_{n+q}\end{bmatrix}$, $\bar{A}_{hv}=\begin{bmatrix}0 & I_{n+q}\\ A_h^e & -E_v^e\end{bmatrix}$, $\bar{B}_h=\begin{bmatrix}0\\ B_h^e\end{bmatrix}$, and $\bar{C}_h=\begin{bmatrix}C_h^e & 0\end{bmatrix}$.

III. Main Results

A. Parameterized LMI conditions

Following the procedure in [21], the observer can be written as

$$\begin{aligned}\bar{E}\dot{\hat{\bar{x}}}&=\bar{A}_{hv}\hat{\bar{x}}+\bar{B}_h u+\bar{L}_{hv}(y-\hat{y})\\ \hat{y}&=\bar{C}_h\hat{\bar{x}},\end{aligned}\tag{7}$$

where $\hat{\bar{x}}=\begin{bmatrix}\hat{x}^e\\ \dot{\hat{x}}^e\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}$ is the estimated state vector and $\bar{L}_{hv}=\begin{bmatrix}0 & L_{hv}^T\end{bmatrix}^T$, $L_{hv}\in\mathbb{R}^{(n+q)\times o}$ is the observer gain to be designed such that $\hat{x}^e\to x^e$ when $t\to\infty$; to this end an extended estimation error is defined:

$$\bar{e}=\bar{x}-\hat{\bar{x}}=\begin{bmatrix}x^e-\hat{x}^e\\ \dot{x}^e-\dot{\hat{x}}^e\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}.\tag{8}$$

Therefore

$$\bar{E}\dot{\bar{e}}=\left(\bar{A}_{hv}-\bar{L}_{hv}\bar{C}_h\right)\bar{e}.\tag{9}$$

Then, consider the Lyapunov function candidate:

$$V(\bar{e})=\bar{e}^T\bar{E}^TP\bar{e};\quad \bar{E}^TP=P^T\bar{E}\geq 0,\tag{10}$$

with $P=\begin{bmatrix}P_1 & 0\\ P_3 & P_4\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}$, $P_1=P_1^T>0$, P_4 being a regular matrix.

The time-derivative of the Lyapunov function gives:

$$\begin{aligned}\dot{V}(\bar{e})&=\dot{\bar{e}}^T\bar{E}^TP\bar{e}+\bar{e}^TP^T\bar{E}\dot{\bar{e}}\\ &=\bar{e}^TP^T\left(\bar{A}_{hv}-\bar{L}_{hv}\bar{C}_h\right)\bar{e}+(*).\end{aligned}\tag{11}$$

Thus $\dot{V}(\bar{e})<0\Leftrightarrow\left(P^T\bar{A}_{hv}-P^T\bar{L}_{hv}\bar{C}_h\right)+(*)<0$ or

$$\begin{bmatrix}P_3^TA_h^e-P_3^TL_{hv}C_h^e+(*) & (*)\\ P_4^TA_h^e-P_4^TL_{hv}C_h^e+P_1-\left(E_v^e\right)^TP_3 & -P_4^TE_v^e+(*)\end{bmatrix}<0.\tag{12}$$

Remark 1. From (12) it is not possible to obtain strict LMI conditions due to the terms $P_3^TL_{hv}$ and $P_4^TL_{hv}$. Several choices are possible that will generate suboptimal solutions. For example, it is possible to obtain parameterized LMI problems by choosing $P_4=\varepsilon P_3$ with ε a scalar and then use a logarithmically spaced search grid [32]–[34]. To get a strict LMI problem following [6] a choice is $P_4=P_3$. In this latter case, with the change of variables: $F_{hv}=P_3^TL_{hv}$, (12) renders:

$$\begin{bmatrix}P_3^TA_h^e-F_{hv}C_h^e+(*) & (*)\\ P_3^TA_h^e-F_{hv}C_h^e+P_1-\left(E_v^e\right)^TP_3 & -P_3^TE_v^e+(*)\end{bmatrix}<0.\tag{13}$$

The following theorem summarizes this result.

Theorem 1. Consider the system (6) together with the observer (7). If there exist matrices $P_1=P_1^T>0$, P_3, and

F_{jk}, $j\in\{1,2,\ldots,r\}$, $k\in\{1,2,\ldots,r_e\}$ such that (1) holds with:

$$\Upsilon_{ij}^{k}=\begin{bmatrix} P_3^T A_i^e - F_{jk}C_i^e + (*) & (*) \\ P_3^T A_i^e - F_{jk}C_i^e + P_1 - \left(E_k^e\right)^T P_3 & -P_3^T E_k^e + (*)\end{bmatrix}, \quad (14)$$

then the estimation error e is asymptotically stable. The observer gains are obtained as $L_{hv}=P_3^{-T}F_{hv}$ and the observer structure is:

$$\begin{aligned} E_v^e\dot{\hat{x}}^e &= A_h^e\hat{x}^e + B_h^e u + L_{hv}\left(y-\hat{y}\right) \\ \hat{y} &= C_h^e\hat{x}^e. \end{aligned} \quad (15)$$

Proof. Based on the developments above, by applying the Relaxation Lemma on (13) gives the desired result. ■

B. Strict LMI conditions

The main result presented hereafter eliminates the drawback that (12) is BMI and does not require choosing $P_3=P_4$. In order to achieve this goal, a new full observer gain needs to be implemented. This is attained by using a new estimated state vector: $\hat{\bar{x}}=\begin{bmatrix}\hat{x}^e \\ \beta^e\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}$.

Then, an observer for (6) writes

$$\bar{E}\dot{\hat{\bar{x}}}=\bar{A}_{hv}\hat{\bar{x}}+\bar{B}_h u+P^{-T}\bar{L}_{hv}\left(y-\hat{y}\right);\quad \hat{y}=\bar{C}_h\hat{\bar{x}}, \quad (16)$$

with $\bar{L}_{hv}=\begin{bmatrix}L_{1hv}^T & L_{2hv}^T\end{bmatrix}^T$, $L_{1hv},L_{2hv}\in\mathbb{R}^{(n+q)\times o}$. Define the estimation error $e=x^e-\hat{x}^e$. The goal is to ensure $e\to 0$ when $t\to\infty$. As previously, we define an extended vector where β^e plays a role equivalent to $\dot{\hat{x}}^e$:

$$\bar{e}=\bar{x}-\hat{\bar{x}}=\begin{bmatrix}x^e-\hat{x}^e \\ \dot{x}^e-\beta^e\end{bmatrix}=\begin{bmatrix}x-\hat{x} \\ d-\hat{d} \\ \hdashline \dot{x}-\beta_1^e \\ \dot{d}-\beta_2^e\end{bmatrix}. \quad (17)$$

Therefore its dynamic is given by

$$\bar{E}\dot{\bar{e}}=\left(\bar{A}_{hv}-P^{-T}\bar{L}_{hv}\bar{C}_h\right)\bar{e}. \quad (18)$$

Consider the Lyapunov function candidate:

$$V\left(\bar{e}\right)=\bar{e}^T\bar{E}^T P\bar{e};\quad \bar{E}^T P=P^T\bar{E}\geq 0, \quad (19)$$

with $P=\begin{bmatrix}P_1 & 0 \\ P_3 & P_4\end{bmatrix}\in\mathbb{R}^{(2n+2q)\times(2n+2q)}$, $P_1=P_1^T>0$, P_4 being a regular matrix. Then, the following result can be stated.

Theorem 2. Consider the system (6) together with the observer (16). If there exist matrices $P_1=P_1^T>0$, P_3, P_4, L_{1jk}, and L_{2jk}, $j\in\{1,2,\ldots,r\}$, $k\in\{1,2,\ldots,r_e\}$ such that (1) holds with

$$\Upsilon_{ij}^{k}=\begin{bmatrix} P_3^T A_i^e - L_{1jk}C_i^e + (*) & (*) \\ P_4^T A_i^e - L_{2jk}C_i^e + P_1 - \left(E_k^e\right)^T P_3 & -P_4^T E_k^e + (*)\end{bmatrix}, \quad (20)$$

then the estimation error e is asymptotically stable and, the observer structure is

$$\begin{aligned} E_v^e\dot{\hat{x}}^e &= A_h^e\hat{x}^e + B_h^e u + \begin{bmatrix}E_v^e & I\end{bmatrix}\left(P^{-1}\right)^T\begin{bmatrix}L_{1hv} \\ L_{2hv}\end{bmatrix}\left(y-\hat{y}\right) \\ \hat{y} &= C_h^e\hat{x}^e. \end{aligned} \quad (21)$$

Proof. The derivative of the Lyapunov function writes:

$$\dot{V}\left(\bar{e}\right)=\bar{e}^T P^T\left(\bar{A}_{hv}-\left(P^{-1}\right)^T\bar{L}_{hv}\bar{C}_h\right)\bar{e}+(*). \quad (22)$$

Then

$$\dot{V}\left(\bar{e}\right)<0 \iff P^T\bar{A}_{hv}-\bar{L}_{hv}\bar{C}_h+(*)<0. \quad (23)$$

Developing (23) gives

$$\begin{bmatrix} P_3^T A_h^e - L_{1hv}C_h^e + (*) & (*) \\ P_4^T A_h^e - L_{2hv}C_h^e + P_1 - \left(E_v^e\right)^T P_3 & -P_4^T E_v^e + (*)\end{bmatrix}<0, \quad (24)$$

which leads to the desired result via the Relaxation Lemma. The final observer form (21) is obtained as follows: recall (16) and define

$$\begin{bmatrix}K_{1hv} \\ K_{2hv}\end{bmatrix}=P^{-T}\bar{L}_{hv}=\begin{bmatrix}P_1^{-1}\left(L_{1hv}-P_3^T P_4^{-T}L_{2hv}\right) \\ P_4^{-T}L_{2hv}\end{bmatrix}. \quad (25)$$

Thus

$$\begin{aligned} \begin{bmatrix}I_{n+q} & 0 \\ 0 & 0_{n+q}\end{bmatrix}\begin{bmatrix}\dot{\hat{x}}^e \\ \dot{\beta}^e\end{bmatrix} &= \begin{bmatrix}0 & I_{n+q} \\ A_h^e & -E_v^e\end{bmatrix}\begin{bmatrix}\hat{x}^e \\ \beta^e\end{bmatrix} \\ &\quad + \begin{bmatrix}0 \\ B_h^e\end{bmatrix}u + \begin{bmatrix}K_{1hv}C_h^e \\ K_{2hv}C_h^e\end{bmatrix}\left(x^e-\hat{x}^e\right), \end{aligned} \quad (26)$$

developing it:

$$\begin{aligned} \dot{\hat{x}}^e &= \beta^e + K_{1hv}C_h^e\left(x^e-\hat{x}^e\right) \\ E_v^e\beta^e &= A_h^e\hat{x}^e + B_h^e u + K_{2hv}C_h^e\left(x^e-\hat{x}^e\right). \end{aligned} \quad (27)$$

By defining $\beta^e=\dot{\hat{x}}^e-K_{1h}C_h^e\left(x^e-\hat{x}^e\right)$, (27) produces:

$$E_v^e\left(\dot{\hat{x}}^e-K_{1hv}\left(y-\hat{y}\right)\right)=A_h^e\hat{x}^e+B_h^e u+K_{2hv}\left(y-\hat{y}\right),$$

which after grouping the terms yields

$$\begin{aligned} E_v^e\dot{\hat{x}}^e &= A_h^e\hat{x}^e + B_h^e u + \left(E_v^e K_{1hv}+K_{2hv}\right)\left(y-\hat{y}\right) \\ &= A_h^e\hat{x}^e + B_h^e u + \begin{bmatrix}E_v^e & I\end{bmatrix}\begin{bmatrix}K_{1hv} \\ K_{2hv}\end{bmatrix}\left(y-\hat{y}\right). \end{aligned} \quad (28)$$

Substituting (25) into (28), the proof is concluded. ■

Note that from (24), one can see that it is possible to add extra degrees of freedom by incorporating convex MFs in matrices P_3 and P_4; thus $P_{3h}=\sum_{j=1}^{r}h_j(z)P_{3j}$ and $P_{4h}=\sum_{j=1}^{r}h_j(z)P_{4j}$, relaxes the conditions in Theorem 2 without augmenting the number of LMIs. This fact is summarized in the next corollary.

Corollary 1. Consider the system (6) together with the observer (16). If there exist matrices $P_1=P_1^T>0$, P_{3j}, P_{4j}, L_{1jk}, and L_{2jk}, $j\in\{1,2,\ldots,r\}$, $k\in\{1,2,\ldots,r_e\}$ such that (1) holds with

$$\Upsilon_{ij}^{k}=\begin{bmatrix} P_{3j}^TA_i^e-L_{1jk}C_i^e+(*) & (*) \\ P_{4j}^TA_i^e-L_{2jk}C_i^e+P_1-\left(E_k^e\right)^TP_{3j} & -P_{4j}^TE_k^e+(*)\end{bmatrix} \tag{29}$$

Then, the estimation error e is asymptotically stable. The observer structure is:

$$\begin{aligned} E_v^e\dot{\hat{x}}^e &= A_h^e\hat{x}^e+B_h^eu+\begin{bmatrix}E_v^e & I\end{bmatrix}P_h^{-T}\begin{bmatrix}L_{1hv}\\ L_{2hv}\end{bmatrix}(y-\hat{y}) \\ \hat{y} &= C_h^e\hat{x}^e. \end{aligned} \tag{30}$$

At last, the strict LMI result in Theorem 1 always encompasses the previous results as stated by the next corollary.

Corollary 2. If the conditions in Theorem 1 are feasible conditions or to the parameterized problem from BMI conditions (12), then there exists a solution to Theorem 2.

Proof. In Theorem 2, expression (20) with $P_4=P_3$ and $L_{1jk}=L_{2jk}=F_{jk}$ renders directly expression (14). For parameterized LMI problems, consider $P_4=\varepsilon P_3$ and $L_{1jk}=L_{2jk}=F_{jk}$ in (20). □

Remark 2. The results given in Theorem 1, Theorem 2, and Corollary 1 can be extended directly to the PI and PMI observers. For a PI observer, set $S=0$, while for the PMI observer consider $d^{(\sigma)}=0$, where σ-derivative of the UI.

Remark 3. The convergence rate of the estimation error can be directly carried out under the TS-LMI framework. In our case, the condition is given by $\dot{V}(\bar{e})\le-2\alpha V(\bar{e})$, $\alpha>0$ [15], [16].

IV. Example

Example 2. Recall the system in Example 1. Considering the compact set $\Omega_x=\{x\in\mathbb{R}^2\}$, the sector nonlinearity approach gives the following TS descriptor model:

$$\begin{aligned} \sum_{k=1}^{2}v_k(z)E_k\dot{x} &= \sum_{i=1}^{2}h_i(z)\left(A_ix+Bu+M_id\right) \\ y &= \sum_{i=1}^{2}h_i(z)\left(C_ix+G_id\right), \end{aligned} \tag{31}$$

where $E_1=\begin{bmatrix}0.87 & -0.17\\ 0.53+\delta & 0.95\end{bmatrix}$, $E_2=\begin{bmatrix}0.87 & 0.83\\ 0.53-\delta & 0.95\end{bmatrix}$, $A_1=\begin{bmatrix}-0.81 & 0.83+\delta\\ -0.74 & 0.57\end{bmatrix}$, $A_2=\begin{bmatrix}-0.81 & 0.83-\delta\\ -0.74 & 0.57\end{bmatrix}$, $B=\begin{bmatrix}0\\ 1\end{bmatrix}$, $M_1=\begin{bmatrix}2 & 1+\delta\\ 1 & -0.5\end{bmatrix}$, $M_2=\begin{bmatrix}2 & 1-\delta\\ 1 & -0.5\end{bmatrix}$, $C_1=\begin{bmatrix}2 & 0\\ 0 & -0.1\end{bmatrix}$, $C_2=\begin{bmatrix}1 & 0\\ 0 & -0.1\end{bmatrix}$, $G_1=\begin{bmatrix}1 & -0.5\\ 0.7 & -0.4\end{bmatrix}$ and $G_2=\begin{bmatrix}1 & -0.5\\ -0.3 & -0.4\end{bmatrix}$. The scalar parameter is $\delta>0$. Consider that x_1 is available. The MFs are $h_1=0.5\left(\cos(x_1)+1\right)$, $h_2=1-h_1$, $v_1=1/\left(1+x_1^2\right)$, and $v_2=1-v_1$. The dynamics of the unknown input are given by the exo-system $\dot{d}=\begin{bmatrix}0 & 0.5\\ -0.5 & 0\end{bmatrix}d$, which generates sinusoidal signals. Note that in [6], [24], $\dot{d}=0$ is considered or that there exists σ so that $d^{(\sigma)}=0$. In order to show the performance of the proposed approaches, two comparisons are done:

1) Comparing with conditions in [24]: In order to use the methodology given in [24], a standard TS representation is needed, thus a TS model with 8 vertices should be obtained after the inversion of the matrix $E(x)$ (see Example 1, Section II.B). By choosing common matrices as follows: $C=C_1$, $G=G_1$, and $S=0$ ($\dot{d}=0$), the following results were obtained: conditions given by Theorem 1 in [24] were feasible until $\delta=0.53$; while conditions in Theorem 1, Theorem 2, and Corollary 1 of this work were feasible up to the value $\delta=0.91$ (the larger δ is the more relaxed the approach is). Therefore, keeping the descriptor structure allows obtaining more relaxed results.

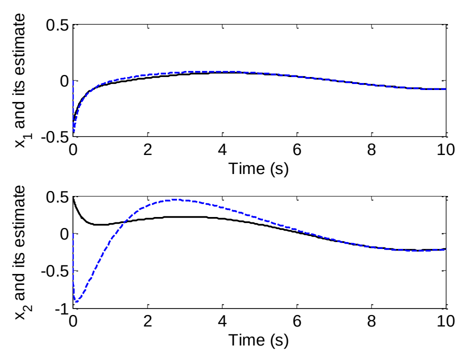

Figure 1. States in (black lines) and their estimates (blue-dashed lines) for Example 1 for *δ=1.55*.

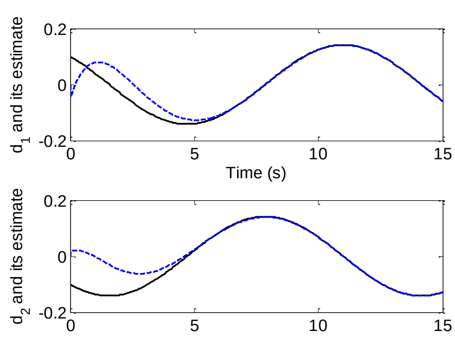

Figure 2. Unknown inputs (black lines) and their estimates (blue-dashed lines) for Example 1 for *δ=1.55*.

2) Comparing the approaches given in Theorems 1 and 2 as well as Corollary 1. The aim is to design an UI observer for the TS descriptor model (31), considering the exo-system. For Theorem 1, the maximum value of δ for which feasible solutions were found is $\delta = 1.16$; in case of Theorem 2, the maximum value of δ for which the conditions were found feasible is $\delta = 1.66$; while for Corollary 1 the maximum δ is $\delta = 1.84$. Thus, Corollary 1 is more relaxed than Theorems 1 and 2.

When considering the exo-system as well as the real parameter $\delta = 1.55$, conditions in Theorem 2 were found feasible. Figures 1 and 2 illustrate the result via simulation with initial conditions $x^e(0) = \begin{bmatrix} -0.4 & 0.5 & 0.1 & -0.1 \end{bmatrix}^T$. Some matrices of the solution are given as example:

$$P_1 = \begin{bmatrix} 0.14 & 0.01 & -1.54 & 0.33 \\ 0.01 & 0.02 & -0.21 & 0.29 \\ -1.54 & -0.21 & 20.96 & -2.79 \\ 0.33 & 0.29 & -2.79 & 22.44 \end{bmatrix}, L_{11}^{(1)} = \begin{bmatrix} 0.39 & -6.49 \\ -0.62 & -4.04 \\ 2.17 & -4.17 \\ -1.92 & -2.21 \end{bmatrix},$$

$$L_{12}^{(1)} = \begin{bmatrix} 0.43 & -5.99 \\ -0.64 & -3.89 \\ 1.93 & -3.74 \\ -1.88 & -2.06 \end{bmatrix}, L_{21}^{(2)} = \begin{bmatrix} -0.32 & -0.55 \\ -0.45 & -0.46 \\ -0.01 & 0.74 \\ -0.81 & -0.59 \end{bmatrix}$$

V. Conclusion

A novel observer for the estimation of unknown inputs has been presented. The approach is based on the Takagi-Sugeno representation of a nonlinear descriptor model. Via an auxiliary state variable, a new observer structure can be designed by means of strict LMI conditions.

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