Control and estimation for mobile sensor-target problems with distance-dependent noise
-full proofs-

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I. MAIN RESULTS

We have:

$$\begin{aligned}\dot{e} =& f_e(e, x_T, \hat{x}_T) - LD\omega(x_T, x_S) + Bd \\ \dot{z} =& f_z(z, x_T, x_S) + BKe + Bd\end{aligned} \tag{1}$$

where

$$\begin{aligned}f_e(e, x_T, \hat{x}_T) :=& (A - LC)e \\ &+ G\left[\psi(Hx_T) - \psi(H\hat{x}_T)\right]\end{aligned} \tag{2}$$

and

$$\begin{aligned}f_z(z, x_T, x_S) :=& (A - BK)z \\ &+ G\left[\psi(Hx_T) - \psi(Hx_S)\right].\end{aligned} \tag{3}$$

$$\dot{\chi} = f(\chi, \eta) + \tilde{D}\omega(x_T, x_S) + \tilde{B}d, \tag{4}$$

where $f(\chi, \eta) = \tilde{A}\chi + \tilde{G}\Delta(t)\eta$.

Theorem 1: Consider system (4) with $d = 0$ and Assumption 2 holds. If matrices $\tilde{P} = \tilde{P}^T > 0$, $R_1 = R_1^T = \text{diag}(r_{11}, ..., r_{1r}) > 0$, $R_2 = \text{diag}(r_{21}, ..., r_{2r}) > 0$, L, K and constant ϵ can be found such that

$$\begin{bmatrix}\tilde{A}^T\tilde{P} + * + \epsilon I & \tilde{P}\tilde{G} + \tilde{H}^T R \\ * & \gamma(R)\end{bmatrix} \leq 0 \tag{5}$$

where $R = \begin{bmatrix}R_1 & 0 \\ 0 & R_2\end{bmatrix}$, $\tilde{H} = \begin{bmatrix}H & 0 \\ 0 & H\end{bmatrix}$, and $\gamma(R) = -2R\,\text{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r}, \frac{1}{b_1}, ..., \frac{1}{b_r})$, then (4) is locally asymptotically stable at the origin. □

Proof: First we prove that in the absence of the ranging noise, ω, the obtained controller and observer gains provide global asymptotic stability at the origin. After which, when ω is considered, we show that the obtained results provide local asymptotic stability. The proof follows the steps of the proof of Theorem 2 from [2].

Let $V(\chi) := \chi^T\tilde{P}\chi$ for any $\chi \in \mathbb{R}^{2n}$. For any $\chi \in \mathbb{R}^{2n}$ and $\eta \in \mathbb{R}^{2r}$,

$$\langle\nabla V(\chi), f(\chi, \eta)\rangle = \begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix}^T \begin{bmatrix}\tilde{A}^T\tilde{P} + \tilde{P}\tilde{A} & \tilde{P}\tilde{G} \\ * & 0\end{bmatrix} \begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix}. \tag{6}$$

By using (5), we have:

$$\begin{aligned}&\langle\nabla V(\chi), f(\chi, \eta)\rangle \leq \begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix}^T \begin{bmatrix}-\epsilon I & -\tilde{H}^T R \\ * & -\gamma(R)\end{bmatrix} \begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix} \\ &= -\epsilon|\chi|^2 - 2\chi^T\tilde{H}^T R\Delta(t)\eta - \eta^T\Delta(t)^T\gamma(R)\Delta(t)\eta.\end{aligned} \tag{7}$$

We examine the term $-2\chi^T\tilde{H}^T R\Delta(t)\eta$:

$$\begin{aligned}&-2\chi^T\tilde{H}^T R\Delta(t)\eta \\ &= -2\begin{bmatrix}e \\ z\end{bmatrix}^T \begin{bmatrix}H^T R_1 & 0 \\ 0 & H^T R_2\end{bmatrix} \begin{bmatrix}\delta_e(t)\eta_e \\ \delta_z(t)\eta_z\end{bmatrix} \\ &= -2e^T H^T R_1\delta_e(t)\eta_e - 2z^T H^T R_2\delta_z(t)\eta_z.\end{aligned} \tag{8}$$

Using $\eta_e = He$, and $\eta_z = Hz$ we obtain for (8):

$$-2\chi^T\tilde{H}^T R\Delta(t)\eta = -2\eta_e^T R_1\delta_e(t)\eta_e - 2\eta_z^T R_2\delta_z(t)\eta_z. \tag{9}$$

Next, the term $-\eta^T\Delta(t)^T\gamma(R)\Delta(t)\eta$ can be rewritten as:

$$\begin{aligned}&-\eta^T\Delta(t)^T\gamma(R)\Delta(t)\eta = \\ &\quad 2(\delta_e(t)\eta_e)^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\delta_e(t)\eta_e \\ &\quad + 2(\delta_z(t)\eta_z)^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\delta_z(t)\eta_z.\end{aligned} \tag{10}$$

This leads to the following:

$$\begin{aligned}&\langle\nabla V(\chi), f(\chi, \eta)\rangle \leq \\ &= -\epsilon|e|^2 - \epsilon|z|^2 - 2\eta_e^T R_1\delta_e(t)\eta_e - 2\eta_z^T R_2\delta_z(t)\eta_z \\ &\quad + 2(\delta_e(t)\eta_e)^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\delta_e(t)\eta_e \\ &\quad + 2(\delta_z(t)\eta_z)^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\delta_z(t)\eta_z.\end{aligned} \tag{11}$$

Now we consider only the terms with η_e, we have:

$$\begin{aligned}&-2\eta_e^T R_1\delta_e(t)\eta_e + 2(\delta_e(t)\eta_e)^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\delta_e(t)\eta_e \\ &= -2\eta_e^T\left(R_1 - \delta_e^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\right)\delta_e(t)\eta_e.\end{aligned} \tag{12}$$

Since R_1, $\delta_e(t)$ are diagonal matrices, we have terms only on the main diagonal, and we can examine each term individually, we have: $r_{1k}(1 - \delta_{ek}\frac{1}{b_k})$. We know that $\delta_{ek} \in [0, b_k]$, and $r_{1k} > 0$, which means that $r_{1k}(1 - \delta_{ek}\frac{1}{b_k}) \geq 0$. This leads to the conclusion that

$$-2\eta_e^T\left(R_1 - \delta_e^T R_1\text{diag}\left[\tfrac{1}{b_1}, ..., \tfrac{1}{b_r}\right]\right)\delta_e(t)\eta_e \leq 0. \tag{13}$$

Similarly, this is true also for the terms with η_z. Finally, we obtain:

$$\langle\nabla V(\chi), f(\chi, \eta)\rangle \leq -\epsilon|\chi|^2, \tag{14}$$

which proves that global asymptotic stability is achieved for (4), when $d = 0$ and $\omega = 0$.

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Next, let us consider the nonlinear uncertain term ω. We want to define the control law so that we can attenuate the effect of this perturbation, i.e. bring the sensor system in the vicinity of the target. We use Assumption 2, and the obtained conditions provide local stability for (4). The term ω can be considered as a vanishing perturbation and we can use Lemma 9.1 from [3]. We have:

$$\begin{aligned}\langle \nabla V(\chi), f(\chi,\eta) + \tilde{D}\omega(x_T,x_S)\rangle = \\ \begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix}^T \begin{bmatrix}\tilde{A}^T\tilde{P}+\tilde{P}\tilde{A} & \tilde{P}\tilde{G} \\ * & 0\end{bmatrix}\begin{bmatrix}\chi \\ \Delta(t)\eta\end{bmatrix} \\ + 2\chi^T\tilde{P}\tilde{D}\omega(x_T,x_S)\end{aligned} \tag{15}$$

Based on (14), we obtain the following:

$$\begin{aligned}\langle \nabla V(\chi), f(\chi,\eta)+\tilde{D}\omega(x_T,x_S)\rangle \leq \\ -\epsilon|\chi|^2 + 2\chi^T\tilde{\tilde{P}}\tilde{D}\omega(x_T,x_S).\end{aligned} \tag{16}$$

Based on Assumption 2, we have: $\omega(x_T,x_S) \leq \zeta_1\|z\|^{\zeta_2}$, and z is included in χ, so the following is true:

$$\omega(x_T,x_S) \leq \zeta_1\|\chi\|^{\zeta_2}. \tag{17}$$

Note that $\zeta_1 > 0$ and $\zeta_2 > 1$, so we can write the following from (16):

$$\begin{aligned}\langle \nabla V(\chi), f(\chi,\eta)+\tilde{D}\omega(x_T,x_S)\rangle \leq \\ -|\chi|^2(\epsilon - 2\zeta_1\|\tilde{P}\tilde{D}\|\|\chi\|^{\zeta_2-1}).\end{aligned} \tag{18}$$

As long as $\epsilon - 2\zeta_1\|\tilde{P}\tilde{D}\|\|\chi\|^{\zeta_2-1} \geq 0$ the following holds:

$$\langle \nabla V(\chi), f(\chi,\eta)+\tilde{D}\omega(x_T,x_S)\rangle < 0. \tag{19}$$

Thus, the origin of (4), is locally asymptotically stable, when $d = 0$. ■

Theorem 2: Consider the plant:

$$\begin{aligned}\dot{z} &= (A-BK)z + G\delta_z(t)\eta_z \\ \eta &= Hz\end{aligned} \tag{20}$$

where $\delta_z(t) = \text{diag}(\delta_{z1},...\delta_{zr})$, and $\forall k \in [1, zr]$, $\delta_{zk} \in [a_k, b_k]$, $0 \leq a_k \leq b_k < \infty$, and define $F = \text{diag}(\frac{1}{b_1},...,\frac{1}{b_r})$. If there exist matrices $P = P^T > 0$, $Q = \text{diag}(q_1,...,q_r) > 0$, N, and a constant $\epsilon > 0$, so that

$$\begin{bmatrix} AP - BN + * & G(FQ)^T + PH^T & P \\ * & -2FQF & 0 \\ * & * & -\frac{1}{\epsilon}I\end{bmatrix} < 0 \tag{21}$$

then the origin of (20) is globally asymptotically stable, and the controller gain can be recovered from $K = NP_2^{-1}$.

Proof: Consider the Lyapunov function $V(z) = z^TP^{-1}z$, and impose that

$$\begin{aligned}\langle \nabla V(z), (A-BK)z + G\delta_z(t)\eta_z\rangle < \\ \begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix}^T \begin{bmatrix} -\epsilon I & -H^T(FQ)^T \\ * & 2Q^{-1}\end{bmatrix}\begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix}.\end{aligned} \tag{22}$$

We have

$$\begin{aligned}&\langle \nabla V(z), (A-BK)z + G\delta_z(t)\eta_z\rangle \\ &= \begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix}^T \begin{bmatrix} P^{-1}(A-BK) + * & P^{-1}G \\ * & 0\end{bmatrix}\begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix},\end{aligned} \tag{23}$$

which gives from (22) the following matrix inequality:

$$\begin{bmatrix} P^{-1}(A-BK) + * + \epsilon I & P^{-1}G + H^T(FQ)^{-T} \\ * & -2Q^{-1}\end{bmatrix} < 0. \tag{24}$$

Congruence with $\begin{bmatrix} P & 0 \\ 0 & FQ\end{bmatrix}$ leads to

$$\begin{bmatrix} (A-BK)P + * + \epsilon P^2 & G(FQ)^T + PH^T \\ * & -2FQF\end{bmatrix} < 0$$

Note that $(FQ)^T = QF$. Applying Schur complement on ϵP^2 and denoting $KP = M$ gives (21). Consider now

$$\begin{aligned}&\begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix}^T \begin{bmatrix} -\epsilon I & -H^T(FQ)^{-T} \\ * & 2Q^{-1}\end{bmatrix}\begin{bmatrix} z \\ \delta_z(t)\eta_z\end{bmatrix} \\ &= -\epsilon z^Tz - 2\eta_z^T\left((FQ)^{-T}\delta_z(t) - \delta_z(t)^TQ^{-1}\delta_z(t)\right)\eta_z\end{aligned}$$

since, all F, Q and δ_z are diagonal, we have

$$(\frac{1}{b_i}q_i)^{-1}\delta_{zi}(t) - \delta_{zi}(t)^2q_i^{-1} = \frac{\delta_{zi}(t)}{q_i}(b_i - \delta_{zi}(t)) \geq 0$$

which leads to:

$$\begin{aligned}&2\eta_z^T\left((FQ)^{-T}\delta_z - \delta_z^TQ^{-1}\delta_z\right)\eta_z \leq 0 \\ \Rightarrow &\langle \nabla V(z), (A-BK)z + G\delta_z(t)\eta_z\rangle \leq -\epsilon\|z\|^2 \leq 0\end{aligned}$$

■

Theorem 3: Consider the plant (1) with $d = 0$ and Assumption 2 holds. If matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $R_1 = R_1^T = \text{diag}(r_{11},...,r_{1r}) > 0$, $Q = Q^T = \text{diag}(q_1,...,q_r) > 0$, M, N and constants ϵ_e and ϵ_z can be found such that:

$$\begin{bmatrix} A^TP_1 - C^TM^T + * + \epsilon_e I & P_1G + H^TR_1 \\ * & \gamma(R_1)\end{bmatrix} \leq 0$$

$$\begin{bmatrix} AP_2 - BN + * & G(FQ)^T + P_2H^T & P_2 \\ * & -2FQF & 0 \\ * & * & -\frac{1}{\epsilon_z}I\end{bmatrix} \leq 0,$$

where $F = \text{diag}(\frac{1}{b_1},...,\frac{1}{b_r})$, and $\gamma(R_1) = -2R_1F$, then the origin of (1) is locally asymptotically stable, and the observer and controller gains can be recovered from $L = P_1^{-1}M$ and $K = NP_2^{-1}$.

Proof: Let, for any $e, z \in \mathbb{R}^n$

$$\begin{aligned}V_e(e) &:= e^TP_1e \\ V_z(z) &:= z^TP_2^{-1}z.\end{aligned} \tag{25}$$

We have

$$\begin{aligned}\langle \nabla V_e(e), f_e(e,x_T,\hat{x}_T)\rangle = e^T\left[(A-LC)^TP_1 + (*)\right]e \\ + 2e^TP_1G\left[\psi(Hx_T) - \psi(H\hat{x}_T)\right],\end{aligned} \tag{26}$$

where f_e is defined in (2). Following the steps of of the proof of Theorem 1 from (8)-(14), we obtain

$$\langle \nabla V_e(e), f_e(e,x_T,\hat{x}_T)\rangle \leq -\epsilon_e\|e\|^2. \tag{27}$$

On the other hand, based on Theorem 2, the LMI

$$\begin{bmatrix} AP_2 - BN + * & G(FQ)^T + P_2H^T & P_2 \\ * & -2FQF & 0 \\ * & * & -\frac{1}{\epsilon}I\end{bmatrix} \leq 0 \tag{28}$$

ensures that

$$\langle \nabla V_z(z), f_z(z, x_T, x_S)\rangle \leq -\epsilon_z \|z\|^2. \tag{29}$$

We proved that (27) and (29) hold for any $e,\, z \in \mathbb{R}^n$. Now based on Section 7.6 from [1], there always exist constants $\lambda,\, \epsilon_{ez} > 0$, for

$$P = \begin{bmatrix} \lambda P_1 & 0 \\ 0 & P_2^{-1} \end{bmatrix} \tag{30}$$

so that

$$\langle \nabla V(e, z), f(e, x_T, \hat{x}_T), f(z, x_T, x_S) + BKe\rangle \\ \leq -\epsilon_{ez}\|\chi\|^2, \tag{31}$$

where ϵ_{ez} depends on ϵ_e, ϵ_z, and P. We obtain global asymptotic stability for (4), when $d = 0$, $\omega = 0$.

When $\omega \neq 0$ we follow the steps from the proof of Theorem 1 from (15) to (18) to ensure local asymptotic stability at the origin of (4), when $d = 0$. ■

Theorem 4: Consider system (4), if matrices $\tilde{P} = \tilde{P}^T > 0$, $R_1 = R_1^T = \text{diag}(r_{11}, ..., r_{1r}) > 0$, $R_2 = \text{diag}(r_{21}, ..., r_{2r}) > 0$, L, K and constants ϵ, μ_d can be found such that

$$\begin{bmatrix} \tilde{A}^T\tilde{P} + * + \epsilon I & \tilde{P}\tilde{G} + \tilde{H}^T R & \tilde{P}\tilde{B} \\ * & \gamma(R) & 0 \\ * & * & -\mu_d I \end{bmatrix} \leq 0 \tag{32}$$

where R, $\tilde{H}$, $\gamma(R)$ are the same as in Theorem 1, then the augmented error dynamics in (4) satisfies

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{D}\omega + \tilde{B}d\rangle - \mu_d\|d\|^2 \leq 0 \tag{33}$$

for all $\|\chi\| \leq \alpha$.

Proof: Let $V(\chi) := \chi^T P\chi$ for any $\chi \in \mathbb{R}^{2n}$, and the derivative along the solutions of (4) will be

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{B}d\rangle = \\ \begin{bmatrix} \chi \\ \Delta(t)\eta \\ d \end{bmatrix}^T \begin{bmatrix} \tilde{A}^T\tilde{P} + * & \tilde{P}\tilde{G} & \tilde{P}\tilde{B} \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} \chi \\ \Delta(t)\eta \\ d \end{bmatrix} \tag{34}$$

Using (32), we obtain:

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{B}d\rangle \leq -\epsilon|\chi|^2 \\ - 2\chi^T\tilde{H}^T R\Delta(t)\eta - \eta^T\Delta(t)^T\gamma(R)\Delta(t)\eta + \mu_d\|d\|^2. \tag{35}$$

It was proved in Theorem 1 that $-2\chi^T\tilde{H}^T R\Delta(t)\eta - \eta^T\Delta(t)^T\gamma(R)\Delta(t)\eta \leq 0$, which leads to:

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{B}d\rangle \leq -\epsilon|\chi|^2 + \mu_d\|d\|^2. \tag{36}$$

Next, we consider also ω, and we obtain:

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{B}d\rangle - \mu_d\|d\|^2 \\ \leq -|\chi|^2(\epsilon - 2\zeta_1\|\tilde{P}\tilde{D}\|\|\chi\|^{\zeta_2 - 1}). \tag{37}$$

There exists a constant $\|\chi\|^2 \leq \alpha$ for which $(\epsilon - 2\zeta_1\|\tilde{P}\tilde{D}\|\|\chi\|^{\zeta_2-1}) \geq 0$, so

$$\langle \nabla V(\chi), f(\chi, \eta) + \tilde{B}d\rangle - \mu_d\|d\|^2 \leq 0 \tag{38}$$

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