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# SATURATED PARTICLE FILTER

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*Abstract*— **In many practical applications the state variables are defined on a compact set of the state space. For estimating such variables constrained particle filters have been successfully applied to nonlinear systems. For the saturated system the measurement information can be used during the sampling procedure to obtain particles that approximate the true state of the system. This can be achieved by using a detection function, which detects the saturation as it occurs. In this paper we propose the Saturated Particle Filter algorithm which incorporates the measurements into the importance sampling procedure through the detection function. The new filter is applied to the Lindley-type stochastic process, where the stochastic process depends on an exogenous parameter. This parameter changes during the simulation. Furthermore, the system is corrupted with high measurement noise. The simulations show that our new filter achieves better performance than the standard Constrained SIR filter, while it preserves low computational complexity.**

## I. INTRODUCTION

Dynamic filters have been studied for decades in various engineering problems which require extracting information of interest from an uncertain or changing environment. Such problems are in general modeled in a *Stochastic Dynamical System* (SDS) framework. When a SDS has linear dynamics and additive Gaussian noises it is well known that the optimal solution, i.e., the estimator that minimizes the mean square error, is given by the *Kalman Filter* (KF) [1]. In case of nonlinear or/and non-Gaussian noises, in general an optimal solution is unknown and one needs to rely on suboptimal ones. Several versions of the KF that give suboptimal solution have been developed to address the nonlinear filtering problem. These include, among others, the Extended KF [1], the Unscented KF [1], [2], [3], the Gaussian Sum KF [1], [4]. These are parametric filters, i.e., filters that solve a finite dimensional estimation problem. Parametric filters perform well when applied to a certain class of models, e.g., stochastic processes that can be accurately approximated by a Gaussian process. However, they cannot be applied to more general systems.

As an alternative to parametric methods, non-parametric filters have been proposed as a tool to solve a general filtering problem. Non-parametric filters aim to estimate a *probability density function* (pdf), thus the problem becomes infinite dimensional. The *Particle Filter* (PF) is one of the most successful non-parametric filters that have been proposed in the filtering community. The PF approximates a pdf of the state of the system by a set of points which are obtained by utilizing the Importance Sampling method [5], and then weighted according to the Bayes rule. However, the PF is based on the Monte Carlo approximation, hence it might require a large number of samples to achieve an accurate estimate. This makes the algorithm computationally expensive, and hence, limits its on-line applicability. The choice of the importance sampling density is a crucial step towards reducing the computational costs, and therefore making the filter feasible for on-line applications.

In this paper we consider processes with saturation, i.e., processes for which at least one of the state variables is defined on a compact set. The point that belongs to the boundary of such a set is called the saturation point. These processes are frequently met in the applied sciences, e.g., in industrial [6], and in the theoretical research [7].

To solve the filtering problem for the continuous-state process with saturation we propose in this paper a novel method to design the importance density.

Design methods for the importance density have been extensively studied [1], [5]. Recently the constrained PF have been proposed [8], [9], [10], [11] which produce a state estimate that does not violate the physical constraints of the system. This is done by discarding unsuitable particles [8], [10], or by projecting them on a constraint region [9], [11]. In the design of our new *Saturated Particle Filter* (Saturated PF), to ensure that the particles are within the permissible region, we use the latter, i.e., the projection approach. We further improve the constrained PF of [9] by introducing a novel sampling method, which effectively detects the saturation moment, and forces the particles to rapidly jump to that part of the state space which is close to the saturation point.

The paper is organized as follows: Section II defines the mathematical framework of the Saturated Stochastic Dynamical System which is the basic object of consideration within this paper. Furthermore, the estimation problem is formulated. In Section III the standard solution to the estimation problem is given. The novel Saturated Particle Filter is derived in Section IV. In Section V the new filter is compared with the filter from Section III. Section VI concludes the paper.

## II. SATURATED STOCHASTIC DYNAMICAL SYSTEM

The goal of this section is to present a mathematical framework which we use to model saturated processes. We first give a general definition of the systems under consideration.

*Definition 1 (Stochastic Dynamical System):*
Assume that for every $k \geq 1$, $w_k$ and $v_k$ are mutually

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independent random variables, $f_k$ is a (possibly nonlinear) function that describes the state evolution, $h_k$ is a (possibly nonlinear) function that establishes the observation model, and $p_0$ is a pdf of the initial state $x_0$. The *Stochastic Dynamical System* (SDS) is defined as a couple $\{(x_k, y_k)\}_{k=0}^{+\infty}$ of discrete-time stochastic processes $\{x_k\}_{k=1}^{+\infty}$, and $\{y_k\}_{k=1}^{+\infty}$ that evolve according to:

$$x_{k+1} = f_k(x_k, w_k), \quad (1)$$
$$y_k = h_k(x_k, v_k), \quad (2)$$
$$x_0 \sim p_0(\cdot). \quad (3)$$

The stochastic process defined by (1)–(3) is a Hidden Markov Model, i.e., given the present state of the system, neither the present observation nor the future state of the system depend on the past states. This property, known as the Markov property [12], allows the estimation of the state of the system recursively, as it is shown in the following sections. To define saturated processes we need the following definition:

*Definition 2 (Saturated Random Variable):* A random variable $\xi$ is *saturated* if there exists a bounded set $A$ such that the probability of $\xi$ belonging to $A$ is equal to one, i.e., $\mathbb{P}(\xi \in A) = 1$.

*Definition 3 (Saturated Stochastic Dynamical System):* Let $\{(x_k, y_k)\}_{k=0}^{+\infty}$ be a SDS defined by (1)–(3). We call the couple $\{(x_k, y_k)\}_{k=0}^{+\infty}$ the *Saturated Stochastic Dynamical Systems* (SSDS) if for each $k \geq 1$, given the state $x_{k-1}$, the state $x_k$ at time $k$, is a saturated random variable.

For simplicity, throughout this paper we assume that $\{x_k\}_{k=1}^{+\infty}$, and $\{y_k\}_{k=1}^{+\infty}$ are one-dimensional real-valued processes[1]. Furthermore, we assume that the process $\{x_k\}_{k=1}^{+\infty}$ is non-negative. In this paper we consider the SSDSs such that for each $k \geq 1$ the upper bound[2] of the variable $x_k$ is dependent only on the past state $x_{k-1}$. More precisely, we consider SSDSs such that the following condition is fulfilled:

*Condition 1 (Saturation Condition):* There exist a function $C : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $\tilde{f}_k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ such that for each $k \geq 1$ (1) takes the form:

$$x_{k+1} = \min\left(\tilde{f}_k(x_k, w_k), C(x_k)\right). \quad (4)$$

The bounds $\{C(x_k)\}_{k=0}^{+\infty}$ of such SSDS form a (possibly unbounded) stochastic process. Possible realization of the stochastic processes $\{x_k\}_{k=1}^{+\infty}$ and $\{C(x_k)\}_{k=0}^{+\infty}$ is illustrated in Figure 1.

We are interested in continuous state space, therefore it is reasonable to assume that for every time step $k$ the random variable $\tilde{f}_k(x_k, w_k)$ has a continuous pdf. This, however, does not hold for the variables $x_k$. Indeed, from (4) it follows that each variable $x_{k+1}$ has a singularity at the point $C(x_k)$.

[1]For the higher dimensional processes the general idea remains the same, but the mathematical derivations become more involved.

[2]Definition 3 allows the bounds for the process $\{x_k\}_{k=1}^{+\infty}$ to vary over the time-steps $k \geq 1$.

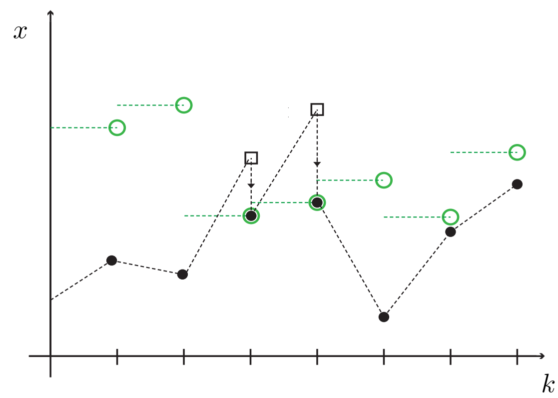


Fig. 1. Trajectories of the saturated process $\{x_k\}_{k=1}^{+\infty}$ (small filled circles) and its bounds $\{C(x_k)\}_{k=0}^{+\infty}$ (large empty circles). When the unsaturated variable $\tilde{f}_k(x_k, w_k)$ (empty squares) exceeds the saturation bound $C(x_k)$ (horizontal dotted lines) it is projected on the appropriate bound (vertical dotted lines). In such cases the realizations of processes $\{x_k\}_{k=1}^{+\infty}$ and $\{C(x_k)\}_{k=0}^{+\infty}$ are overlapping (small circles within large circles).

This means that the pdf of $x_{k+1}$ is continuous up to the point $C(x_k)$ in which the positive probability mass is focused. Therefore, the conditional density of the variable $x_{k+1}$ given the previous state $x_k$ is given by:

$$\mathbb{P}(x_{k+1} = x | x_k) = \mathbb{P}\left(\tilde{f}_k(x_k, w_k) = x | x_k\right) \mathbf{1}_{[0, C(x_k))}(x) \quad (5a)$$
$$+ \int_{C(x_k)}^{+\infty} \mathbb{P}\left(\tilde{f}_k(x_k, w_k) = z | x_k\right) dz \, \delta_{C(x_k)}(x), \quad (5b)$$

where $\mathbf{1}_{[0, C(x_k))}$ is an indicator function on the interval $[0, C(x_k))$, and $\delta_{C(x_k)}$ is a Dirac delta centered at the point $C(x_k)$. The pdf of such a variable is illustrated on Figure 2.

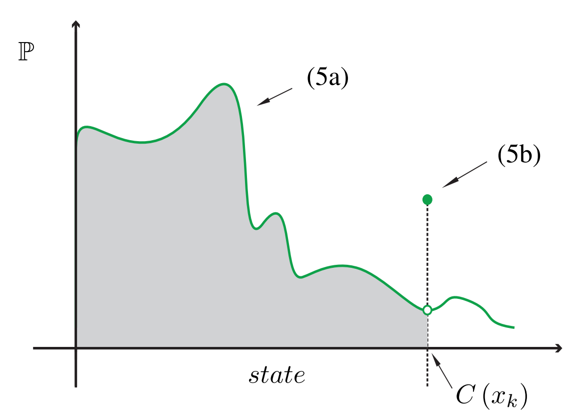


Fig. 2. The pdf of the saturated variable $x_{k+1}$ given the past state $x_k$. The pdf is composed of a continuous part (5a) and a singular mass (5b) concentrated at the saturation point $C(x_k)$.

Having the SSDS defined in such a way, we are interested in estimating the actual state $x_k$ of the system from the available measurements $y_k$. The next section describes a standard estimation method which is applicable to a wide range of the SDSs, including the SSDSs.

## III. SIR PARTICLE FILTER

The Markovian character of the SSDS makes it possible, for estimation purposes, to employ recursive algorithms utilizing Bayes theorem. Since the SSDS is, in general, a nonlinear and non-Gaussian system, it is suggested to use the PF in order to get accurate estimates [5].

Every time a measurement $y_k$ is obtained, the PF combines it with the previous estimate and returns the estimate of the pdf of the current state of the system $\mathbb{P}(x_k = x|y_k)$. This is achieved in two steps:

1) *Prediction:* the estimate of the pdf of the most recent state of the system $\mathbb{P}(x_{k-1} = x|y_{k-1})$ is propagated through the state-transition model (1) one step ahead. As a result the *predicted density* $\mathbb{P}(x_k = x|y_{k-1})$ is obtained.
2) *Update:* the predicted density is compared with the measurement $y_k$, and then transformed according to the Bayes rule. The final output of the estimation is the *updated density* $\mathbb{P}(x_k = x|y_k)$.

The PF is a Monte Carlo-type algorithm which represents the estimated pdf by the set of $N$ pairs $\left\{\left(x_k^i, \omega_k^i\right)\right\}_{i=1}^N$ of particles ($x_k^i$) and associated weights ($\omega_k^i$). These pairs approximate the true pdf by the formula:

$$\mathbb{P}(x_k = x|y_k) \approx \sum_{i=1}^{N} \omega_k^i \delta_0\left(x - x_k^i\right). \tag{6}$$

The set of particles and weights is obtained in the following manner:

1) At time step $k-1$ the pdf $\mathbb{P}(x_{k-1} = x|y_{k-1})$ is represented by the set $\left\{\left(x_{k-1}^i, \omega_{k-1}^i\right)\right\}_{i=1}^N$,
2) when the measurement $y_k$ becomes available new particles $x_k^i$ are drawn from the *importance density function* (idf) $\mathbb{Q}\left(\cdot|x_{k-1}^i, y_k\right)$,
3) the weights $\omega_{k-1}^i$ are updated through the formula:

$$\tilde{\omega}_k^i = \omega_{k-1}^i \frac{\mathbb{P}\left(h_k\left(x_k, v_k\right) = y_k|x_k = x_k^i\right)\mathbb{P}\left(x_k = x_k^i|x_{k-1}^i\right)}{\mathbb{Q}\left(x_k^i|x_{k-1}^i, y_k\right)}, \tag{7}$$

4) the weights $\omega_k^i$ are obtained by normalizing $\tilde{\omega}_k^i$:

$$\omega_k^i = \frac{\tilde{\omega}_k^i}{\sum_{j=1}^N \tilde{\omega}_k^j}. \tag{8}$$

The problem of such a recursive algorithm is the *particle degeneracy*: after several iterations the whole probability mass is focused on a few particles, whereas all the remaining particles have negligible weights. When this phenomenon occurs, the estimation accuracy degrades. To overcome this problem, a resampling procedure is used. The idea is as follows: at each iteration the degeneracy measure, called *effective sample size* [5], is computed:

$$N_{\text{eff}} = \frac{1}{\sum_{i=1}^N \left(\omega_k^i\right)^2}. \tag{9}$$

When $N_{\text{eff}}$ drops below a specified threshold $N_T \in [1, N]$, particles are resampled using a specific algorithm.

There are many variations of PFs [5], which employ various importance densities and resampling algorithms. To solve the estimation problem for the saturated process we used the *Constrained Sampling Importance Resampling* (Constrained SIR) filter, i.e., the SIR filter [5] modified by the projection algorithm from [9]. In the SIR algorithm the importance density is chosen to be the transition density:

$$\mathbb{Q}\left(x|x_{k-1}^i, y_k\right) := \mathbb{P}\left(x_k = x|x_{k-1} = x_{k-1}^i\right), \tag{10}$$

and the resampling is performed as described in Algorithm 1.

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**Algorithm 1** SIR Resampling

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**Require:** $\{(x^i, w^i)\}_{i=1}^N$
**Ensure:** $\{(x^i_{\text{new}}, w^i_{\text{new}})\}_{i=1}^N$
  **for** $i = 1, 2, \ldots, N$ **do**
    Compute cumulative sum of weights: $w_c^i = \sum_{j=1}^i w_k^j$
  **end for**
  Draw $u_1$ from the uniform distribution $\mathcal{U}(0, \frac{1}{N})$
  **for** $i = 1, 2, \ldots, N$ **do**
    Find $x^{+i}$, the first sample for which $w_c^i \geq u_i$.
    Replace particle $i$: $x^i_{\text{new}} = x^{+i}$, $w^i_{\text{new}} = \frac{1}{N}$
    $u_{i+1} = u_i + \frac{1}{N}$
  **end for**

---

In the SIR framework, because of (10), the weight update (7) is simplified to:

$$\tilde{\omega}_k^i = \omega_{k-1}^i \mathbb{P}\left(h_k\left(x_k, v_k\right) = y_k|x_k = x_k^i\right). \tag{11}$$

However, with such a choice of the importance density, the most recent information $y_k$ is not used during the particle drawing. This information can be of crucial importance in case of saturated processes, thus its loss is undesirable. Therefore, in the next section we derive a new PF that uses the importance density which accounts for the latest measurement $y_k$. The resampling procedure for the new filter is performed by Algorithm 1.

## IV. SATURATED PARTICLE FILTER

In this section we propose a new Saturated PF that is designed for the saturated processes. We begin with the following definition:

*Definition 4 (Detection function):* The function $\alpha : \mathbb{R} \to \mathbb{R}$ is called a *detection function* if the following conditions are fulfilled:

1) there exists $c \in \mathbb{R}$ such that $\alpha(c) = 0$,
2) $\alpha$ is non-decreasing

The purpose of the detection function, as it is shown in what follows, is to quickly detect that the saturation occurred by comparing the measurements with the state constraints. This information is used to force the particles to move to the appropriate region.

Let us consider the SSDS defined by (1)–(4). Furthermore, let $\left\{\left(x_k^i, \omega_k^i\right)\right\}_{i=1}^N$ be the approximation of the updated density of that process at time step $k$. For each $i \in \{1, ..., N\}$, given the previous particle $x_k^i$, the probability that the particle $x_{k+1}^i$ will be saturated[3] follows by (5b):

$$\mathbb{P}\left(x_{k+1}^i = C\left(x_k^i\right)\right) = \int_{C\left(x_k^i\right)}^{+\infty} \mathbb{P}\left(\tilde{f}_k\left(x_k, w_k\right) = z | x_k^i\right) dz. \tag{12}$$

For the ease of notation the right-hand side of (12) is denoted as $q_i$, i.e.,

$$q_i = \int_{C\left(x_k^i\right)}^{+\infty} \mathbb{P}\left(\tilde{f}_k\left(x_k, w_k\right) = z | x_k^i\right) dz. \tag{13}$$

Since the probability $q_i$ depends only on the previous state $x_k^i$, we call it *the predicted probability of saturation*.

Let $\alpha$ be a given detection function satisfying Definition 4. Furthermore, assume that the measurement $y_{k+1}$ becomes available. Then, for each $i \in \{1, ..., N\}$ we define $q_i^\alpha$:

$$q_i^\alpha := \min\left(\max\left[q_i + \alpha\left(y_{k+1} - C\left(x_k^i\right)\right), 0\right], 1\right) \tag{14}$$

The so defined $q_i^\alpha$ depends on both the last state $x_k^i$, and the latest measurement $y_{k+1}$. Therefore, we call it *the updated probability of saturation*.

Using $q_i^\alpha$ defined in (14), and the detection function $\alpha$, we define the importance density $\mathbb{Q}^\alpha$ of the new PF by:

$$\mathbb{Q}^\alpha\left(x | x_k^i, y_{k+1}\right) := q_i^\alpha \delta_{C\left(x_k^i\right)}(x) \tag{15a}$$

$$+ \frac{1 - q_i^\alpha}{1 - q_i} \mathbb{P}\left(\tilde{f}_k\left(x_k, w_k\right) = x | x_k^i\right) \mathbf{1}_{\left[0, C\left(x_k^i\right)\right)}(x). \tag{15b}$$

It can be easily seen that $\mathbb{Q}^\alpha$ defines a probability measure[4]. The importance density of the Constrained SIR filter is a special case of $\mathbb{Q}^\alpha$ with $\alpha \equiv 0$.

Given the particle $x_k^i$, a new particle $x_{k+1}^i$ is drawn from the importance density $\mathbb{Q}^\alpha$. According to (15a) the particle $x_{k+1}^i$ is saturated with the probability $q_i^\alpha$, and with probability $1 - q_i^\alpha$ it is drawn from (15b). The associated weights $\omega_{k+1}^i$ are computed using (7). If $x_{k+1}^i$ saturates, i.e., $x_{k+1}^i = C\left(x_k^i\right)$, then, by the definitions of $q_i$, and $q_i^\alpha$, the weight $\omega_{k+1}^i$ follows the formula:

$$\omega_{k+1}^i \propto \omega_k^i \frac{q_i}{q_i^\alpha} \mathbb{P}\left(h_{k+1}\left(x_{k+1}, v_{k+1}\right) = y_{k+1} | x_{k+1}^i\right), \tag{16}$$

if $x_{k+1}^i$ does not saturate, the weight $\omega_{k+1}^i$ is updated by:

$$\omega_{k+1}^i \propto \omega_k^i \frac{1 - q_i}{1 - q_i^\alpha} \mathbb{P}\left(h_{k+1}\left(x_{k+1}, v_{k+1}\right) = y_{k+1} | x_{k+1}^i\right). \tag{17}$$

The new PF is summarized in Algorithm 2.

The proposed Saturated PF combines the previous states $x_k^i$s with the most recent measurement $y_{k+1}$ to compute the updated probability of saturation $q_i^\alpha$. For large values of $q_i^\alpha$ the algorithm forces the particles to be close to the saturation region[5], whereas for small values of $q_i^\alpha$ the particles are set further from the saturation region. Figure 3 presents the difference between the Unconstrained SIR sampling, the Constrained SIR sampling and the Saturated PF sampling for a large value of $q_i^\alpha$.

The accuracy of the estimation depends on the detection function, which must be chosen appropriately to the SSDS under consideration.

**Algorithm 2** Saturated PF with detection function $\alpha$

**Require:** $\left\{\left(x_k^i, \omega_k^i\right)\right\}_{i=1}^N$
**Ensure:** $\left\{\left(x_{k+1}^i, \omega_{k+1}^i\right)\right\}_{i=1}^N$
**for** $i = 1, 2, \ldots, N$ **do**
  Compute the probability $q_i$ according to (13)
  Compute the probability $q_i^\alpha$ according to (14)
  Draw $u \sim \mathcal{U}(0, 1)$
  **if** $u \leq q_i^\alpha$ **then**
    Particle $x_{k+1}^i$ saturates:

$$x_{k+1}^i := C\left(x_k^i\right)$$
$$\omega_{k+1}^i \propto \omega_k^i \frac{q_i}{q_i^\alpha} \mathbb{P}\left(h_{k+1}\left(x_{k+1}, v_{k+1}\right) = y_{k+1} | x_{k+1}^i\right)$$

  **else**
    Particle $x_{k+1}^i$ does not saturate:

$$x_{k+1}^i \sim \frac{1 - q_i^\alpha}{1 - q_i} \mathbb{P}\left(\tilde{f}_k\left(x_k, w_k\right) = \bullet | x_k^i\right) \mathbf{1}_{\left[0, C\left(x_k^i\right)\right)}(\bullet)$$
$$\omega_{k+1}^i \propto \omega_k^i \frac{1 - q_i}{1 - q_i^\alpha} \mathbb{P}\left(h_{k+1}\left(x_{k+1}, v_{k+1}\right) = y_{k+1} | x_{k+1}^i\right)$$

  **end if**
**end for**

## V. APPLICATION

In this section we apply the Saturated PF to a system which depends on an external parameter $\theta$, and allows relatively large measurement noises. We show that with the proper choice of the detection function $\alpha$, the Saturated PF outperforms the Constrained SIR filter in tracking rapid changes in the dynamics of the system.

The process used to compare the Saturated PF and the Constrained SIR filter to the SSDS given by:

$$x_{k+1} = \min\left(x_k + w_k, C\left(x_k\right)\right), \tag{18}$$
$$y_k = x_k + v_k, \tag{19}$$

where $w_k$ is an exponential random variable with parameter $\theta \cdot C\left(x_k\right)$, i.e., with the expected value $\mathbb{E}w_k = \left(\theta \cdot C\left(x_k\right)\right)^{-1}$. The variable $v_k$ is a zero-mean Gaussian variable with the standard deviation $\sigma_v$.

The state model (18) is nonlinear and non-Gaussian, whereas the observation model (19) is both linear, and conditionally Gaussian. The stochastic process (18) is a Lindley-type process, i.e., it is a modification of the celebrated *Lindley's recursion*, one of the most studied stochastic

[3]The particle $x_{k+1}^i$ is saturated means that $x_{k+1}^i$ is projected on $C\left(x_k^i\right)$ which is equivalent to the projection method described in [9]. Indeed, it makes no difference whether the 'bad' particles drawn from an unconstrained continuous distribution are projected on the saturation point, or each particle is set to saturation point with the probability of saturation. The resulting sets of particles are equivalent in the statistical sense.

[4]$\mathbb{Q}^\alpha$ is positive, and it integrates to one.

[5]The value of saturation $C(x)$ is a random variable dependent on $x$, where $x$ is approximated by $\left\{\left(x^i, \omega^i\right)\right\}$. Therefore, by the saturation region we understand the set $\left\{C\left(x^i\right)\right\}$.

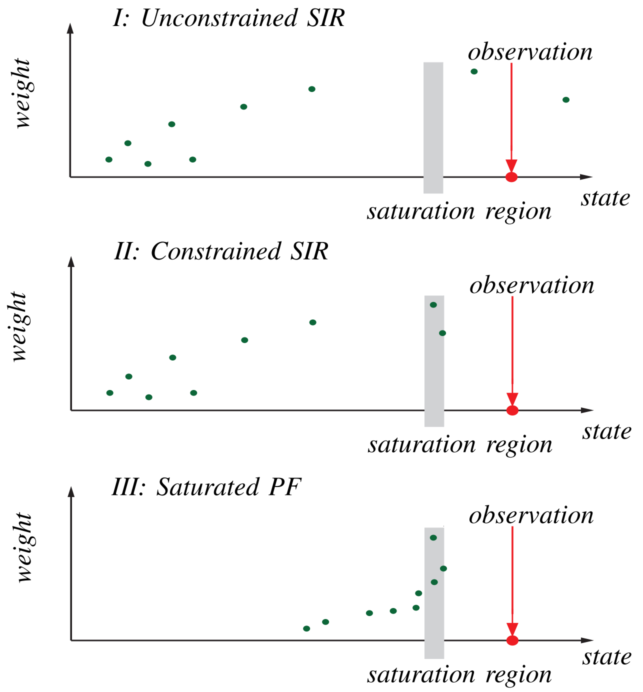


Fig. 3. Distribution of particles obtained by the Unconstrained SIR (top), Constrained SIR (middle) and the Saturated PF (bottom). Some of the particles obtained by the Unconstrained SIR violate the physical constraints on the system (saturation region), other are located far from the actual measurement. The Constrained SIR projects the unphysical particles onto the saturation region, but does not move the remaining particles. The Saturated PF projects the bad particles on the saturation region and forces the remaining particles to concentrate close to saturation region.

processes in applied probability [7], [13]. These type of processes are extensively used in queueing theory [13], [14].

The boundary function $C(\cdot)$ is defined by:

$$C(x) := \begin{cases} x+4 & \textbf{if} \quad x < 15, \\ 0.7x + 8.5 & \textbf{otherwise}. \end{cases} \tag{20}$$

To illustrate the capabilities of the proposed filter, starting from the initial state of the system $x_0 = 7$, we simulated the evolution of the system (18)–(19) for 100 time steps. During the first 50 steps, parameter $\theta$ is set to 1, during the second 50 steps it is set to $\frac{1}{30}$. This models a rapid change in the conditions external to the system.

To simulate a noisy-measurement environment (19), the standard deviation $\sigma_v$ of the variable $v_k$ is set to $\sigma_v = 3$.

Figures 4 and 5 present two independent simulation runs of the system (18)–(19) and two filtered signals (Constrained SIR and Saturated PF). Both Constrained SIR and Saturated PF use the state model (18) with parameter $\theta = 1$ for the whole time of the simulation. The initial state $p_0$ for both filters is equal to $p_0(\cdot) = \mathcal{N}(\cdot; 7, 1)$ (the pdf of the Gaussian variable with the mean and the standard deviation equal to 7 and 1, respectively). The number of particles is set to $N = 100$, and the resampling threshold is set to $N_T = 50$. Figure 4 presents the Saturated PF with the antisymmetric detection function $\alpha_1$, whereas Figure 5 presents the Saturated PF that uses the asymmetric detection function $\alpha_2$.

The Saturated PF from Figure 4 uses an detection function $\alpha_1$ defined as:

$$\alpha_1(x) := \begin{cases} \log(x+1) & \textbf{if} \quad x > 0, \\ -\log(-x+1) & \textbf{otherwise}. \end{cases} \tag{21}$$

Function $\alpha_1$ is antisymmetric in zero, which means that the probability of saturation $q_i^\alpha$ is increased or decreased proportionally to the distance between the measurement $y_{k+1}$ and the saturation bound $C\left(x_k^i\right)$. If the distance $|y_{k+1} - C\left(x_k^i\right)|$ is greater than $(\approx)$ 1.7 then, depending on the sign of the difference, the probability of saturation $q_i^\alpha$ is equal to zero or to one.

The Saturated PF from Figure 5 uses an detection function $\alpha_2$ defined as:

$$\alpha_2(x) := \begin{cases} \log(x+1) & \textbf{if} \quad x > 0, \\ -\log(-x+1) & \textbf{if} \quad x > -\frac{1}{2}, \\ -3\log(-x+1) + 2\log\left(\frac{3}{2}\right) & \textbf{otherwise}. \end{cases} \tag{22}$$

Function $\alpha_2$ is not antisymmetric as was $\alpha_1$. In this case, when the measurement $y_{k+1}$ is smaller than $C\left(x_k^i\right) - \frac{1}{2}$, the probability of saturation $q_i^\alpha$ decreases much faster with the distance $|y_{k+1} - C\left(x_k^i\right)|$ and reaches zero when $y_{k+1} < C\left(x_k^i\right) - 0.83$. When the measurement $y_{k+1}$ is greater than $C\left(x_k^i\right) - \frac{1}{2}$ the probability of saturation $q_i^\alpha$ is adjusted identically as it was for the function $\alpha_1$.

The estimated signals from Figures 4 and 5 are computed as the average of ten independent filter runs. In each of the parallel runs, for both Constrained SIR and Saturated PF, the estimated value of the state is computed by taking a weighted mean of the particles, i.e., $\hat{x}_k = \sum_{i=1}^N \omega_k^i x_k^i$. This corresponds to the Minimum Mean Square Error (MMSE) estimator [1].

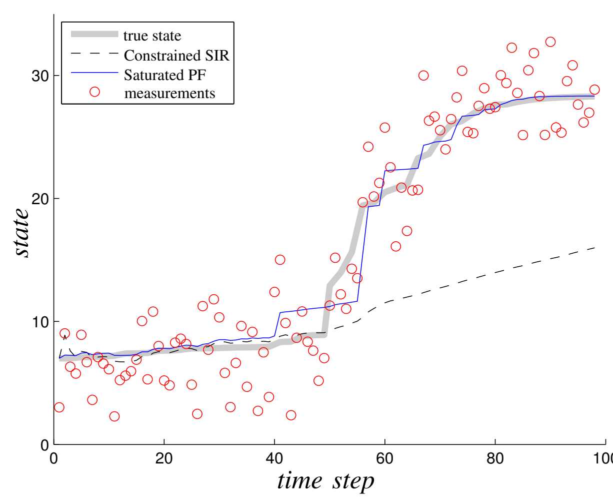


Fig. 4. Constrained SIR and Saturated PF applied to system (18)–(19). The thick solid line is the true value of the state, the circles denote the measurements of the system, the thin solid line represents the MMSE estimate of the state obtained by the Saturated PF with detection function $\alpha_1$, and the thin dashed line denotes the MMSE estimate obtained by the Constrained SIR filter.

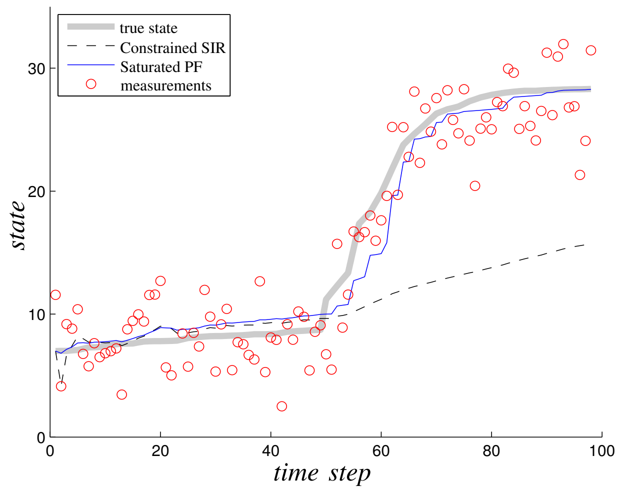


Fig. 5. Constrained SIR and Saturated PF applied to system (18)–(19). The thick solid line is the true value of the state, the circles denote the measurements of the system, the thin solid line represents the MMSE estimate of the state obtained by the Saturated PF with detection function $\alpha_2$, and the thin dashed line denotes the MMSE estimate obtained by the Constrained SIR filter.

The results presented in Figures 4 and 5 show that both the Saturated PF and the Constrained SIR filter perform similarly during the phase when their state model corresponds with the true state process ($\theta = 1$). When the external parameter changes ($\theta = \frac{1}{30}$) the Saturated PF is able to track the true state, whereas the Constrained SIR filter fails to do so. The difference in detection functions, $\alpha_1$ and $\alpha_2$ does not result in a qualitative change of filtered signal.

## VI. CONCLUSIONS

In this paper we proposed a novel filtering method which makes an effective use of the measurements when sampling particles within the particle filter framework. The Saturated PF is designed for a class of SSDSs. The filter makes use of a detection function to detect the saturation of the process. As demonstrated by simulations of the noisy-measurement system, the Saturated PF outperforms the standard PF method in terms of accuracy of tracking the signal that exhibit rapid changes in the dynamics. While better performance is achieved, the computational complexity of the new filter is comparable to the complexity of the Constrained SIR filter.

In general, the accuracy of the estimation depends on the appropriate choice of the detection function. This issue will be addressed in our further research.

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