Fuzzy models and observers for freeway traffic state tracking

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***Abstract*— Traffic state estimation is a prerequisite for traffic surveillance and control. For macroscopic traffic flow models several estimation methods have been investigated, including extended and unscented Kalman filters and particle filters. In this paper we propose a fuzzy observer for the continuous time version of the macroscopic traffic flow model METANET. In order to design the observer, we first derive a dynamic Takagi-Sugeno fuzzy model that exactly represents the traffic model of a segment of a highway stretch. The fuzzy observer is designed based on the fuzzy model and applied to the traffic model. The simulation results are promising for the future development of fuzzy observers for a highway stretch or a whole traffic network.**

I. Introduction

Reliable traffic models are important for several tasks: simulation of different scenarios and conditions, prediction of traffic conditions that will occur in a network, and traffic control. The modeling of traffic networks may be performed at several levels: microscopic, mesoscopic, or macroscopic level. Traffic state estimation is a fundamental task for traffic surveillance and control [1], [11], [17], and has been extensively investigated during the last decades [4], [9], [16]. Vehicular traffic in general has a highly nonlinear behavior, and for freeway networks or stretches it is often described by macroscopic models. These models represent the average traffic behavior in aggregated variables (average flow, density, and speed on a segment of a highway). Some of these variables are measured by various detectors, e.g., loop detectors, video cameras, radars. However, the measurements are in general corrupted by noise, or data may be missing, thereby making the estimation or filtering of these variables necessary.

One of the intensively used models for traffic state estimation is the second-order macroscopic traffic flow model METANET [15], which describes a stretch of a highway in terms of flow, density, and average speed. Several estimation methods, both online and offline have been applied based on this model: Extended Kalman filters [7], [21], [22], Unscented Kalman filters [7], [12], and more recently, particle filters [12]. These filters are applied to a discrete traffic flow model, where the model mismatch is accounted for by noise. The nonlinear versions of the Kalman filter approximate the distributions of the variables with Gaussians. However, since the model is highly nonlinear, the transformation of Gaussian random variables is no longer Gaussian. Therefore, to obtain more accurate estimates of the variables, particle filters have been used. However, they have the disadvantage of large computational costs.

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To obtain more accurate estimates with reduced computational costs, in this paper, we propose a fuzzy observer for freeway traffic state estimation in continuous time. A large class of nonlinear systems can be represented or well approximated by Takagi-Sugeno (TS) fuzzy models [18], which in theory can approximate a general nonlinear system to an arbitrary degree of accuracy [5]. The TS fuzzy model consists of a fuzzy rule base. The rule antecedents partition a given subspace of the model variables into fuzzy regions, while the consequent of each rule is usually a linear or affine model, valid locally in the corresponding region.

For a TS fuzzy model, well-established methods and algorithms can be used to design observers that estimate unmeasurable states. Several types of such observers have been developed for TS fuzzy systems, among which: fuzzy Thau-Luenberger observers [19], [20], reduced-order observers [2], [3], and sliding-mode observers [14]. In general, the design methods for observers lead to an Linear Matrix Inequality (LMI) feasibility problem, which is easy to solve.

In this paper, we aim to develop a fuzzy representation of the second-order macroscopic flow model, and design observers for the obtained fuzzy model. Although fuzzy methods have been used for traffic modeling and control, and set-valued methods have been applied to estimate the traffic density [10], to the authors' knowledge, TS fuzzy observers have not been used for traffic state tracking.

The rest of the paper is organized as follows. The traffic flow model in details and the variables and parameters used are described in Section II. The derivation of the fuzzy models is presented in Section III. The observer design for these fuzzy models, together with simulation results, are presented in Section IV. Finally, Section V concludes the paper.

II. The traffic model

In this paper, we consider a macroscopic model of traffic flow, namely the METANET model developed in [15]. In this model, a freeway link m is divided into N_m segments (see Figure 1), each with a length L_m and λ_m lanes. Traffic dynamics are described in terms of space-mean speed, $v_{m,i}$, flow, $q_{m,i}$, and density, $\rho_{m,i}$, where i denotes the segment index. This model is in general used in discrete time, with a sampling period T. The model equations are derived from the fundamental relation between speed, density, and flow, the law of conservation of vehicles, and a heuristic relationship

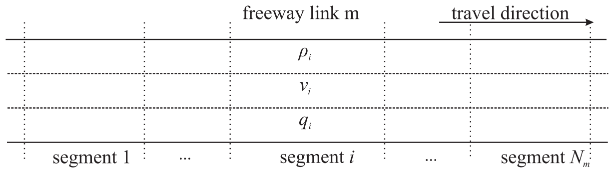

Fig. 1. Traffic variables in the traffic flow model.

of the speed dynamics. The basic equations[1] that describe each segment i of a link m are given as [9]:

$$\begin{aligned}
\rho_{m,i}(k+1) &= \rho_{m,i}(k) + \frac{T}{L_m\lambda_m}(q_{m,i-1}(k) - q_{m,i}(k)) \\
q_{m,i}(k) &= \rho_{m,i}(k)\cdot v_{m,i}(k)\cdot\lambda_m \\
v_{m,i}(k+1) &= v_{m,i}(k) + \frac{T}{\tau}\Big[V(\rho_{m,i}(k)) - v_{m,i}(k)\Big] \\
&\quad + \frac{T}{L_m} v_{m,i}(k)(v_{m,i-1}(k) - v_{m,i}(k)) \\
&\quad - \frac{\nu\cdot T}{\tau\cdot L_m}\frac{\rho_{m,i+1}(k) - \rho_{m,i}(k)}{\rho_{m,i}(k)+\kappa} + \xi_v(k) \\
V(\rho_{m,i}(k)) &= v_{f,m}\cdot\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_{m,i}(k)}{\rho_{cr,m}}\Big)^{a_m}\Big]
\end{aligned} \tag{1}$$

where ξ_v is a random variable to account for the model mismatch in the speed dynamics.

The model (1) can be seen as a simple Euler discretization of the continuous-time model

$$\begin{aligned}
\dot\rho_{m,i} &= \frac{1}{L_m\lambda_m}(q_{m,i-1} - q_{m,i}) \\
q_{m,i} &= \rho_{m,i}\cdot v_{m,i}\cdot\lambda_m \\
\dot v_{m,i} &= \frac{1}{\tau}\Big[V(\rho_{m,i}) - v_{m,i}\Big] + \frac{1}{L_m}v_{m,i}(v_{m,i-1} - v_{m,i}) \\
&\quad - \frac{\nu}{\tau\cdot L_m}\frac{\rho_{m,i+1} - \rho_{m,i}}{\rho_{m,i}+\kappa} + d\xi_v \\
V(\rho_{m,i}) &= v_{f,m}\cdot\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_{m,i}}{\rho_{cr,m}}\Big)^{a_m}\Big]
\end{aligned} \tag{2}$$

where $d\xi_v$ denotes the continuous-time disturbance.

Measured variables are in general the traffic flow $q_{m,i}$, and the mean speed $v_{m,i}$, therefore the measurements y_1 and y_2 are

$$y_1 = q_{m,i} \qquad y_2 = v_{m,i} \tag{3}$$

These measurements are usually corrupted by noise, i.e., a correct expression is

$$y_1 = q_{m,i} + \zeta_q \qquad y_2 = v_{m,i} + \zeta_v \tag{4}$$

The variable that has to be estimated on-line is $\rho_{m,i}$, the traffic density on the ith segment.

In the sequel, model (2) is used and only one link is considered, therefore the subscript m is dropped. The variables are defined in Table I. The parameters and their values, adapted from [9], that are used for simulation purposes, are defined in Table II.

[1]In this paper, we only consider the model (1). Note however, that other equations, such as node, on-ramp, off-ramp, merge, split can also be dealt with.

TABLE I
VARIABLES IN THE TRAFFIC MODEL.

Symbol	Variable	Units
k	time step	–
i	segment index	–
$\rho_{m,i}(k)$	traffic density	veh/km/lane
$v_{m,i}(k)$	space-mean speed	km/h
$q_{m,i}(k)$	traffic volume or flow	veh/h

TABLE II
PARAMETERS OF THE TRAFFIC MODEL.

Symbol	Parameter	Value	Units
L_m	length of segment	0.5	km
λ_m	number of lanes	3	–
$v_{\mathrm{f},m}$	free flow speed	102	km/h
$\rho_{\mathrm{cr},m}$	critical density	30	veh/km/lane
τ	time constant	18	s
ν	anticipation constant	60	km^2/h
κ	constant	40	veh/km
a_m	parameter	2.34	–
$v_{\min}$	minimum velocity	7.4	km/h
$v_{\max}$	maximum velocity	200	km/h
$\rho_{\min}$	minimum density	0	veh/km/lane
$\rho_{\max}$	maximum density	150	veh/km/lane

III. FUZZY TRAFFIC MODELS

A. State-space description

In order to better illustrate how the fuzzy models are derived, consider one segment, and assume that both the state transition model (2) and the measurement model (3) are exact, i.e, the models are not corrupted by noise. Noise will be considered in the observer design in Section IV-C.

Then, the nonlinear system for which a fuzzy model should be obtained is

$$\begin{aligned}
\dot\rho_i &= \frac{1}{L\lambda}(q_{i-1} - q_i) \\
q_i &= \rho_i\cdot v_i\cdot\lambda \\
\dot v_i &= \frac{1}{\tau}\Big[V(\rho_i) - v_i\Big] + \frac{1}{L}v_i(v_{i-1} - v_i) \\
&\quad - \frac{\nu}{\tau\cdot L}\frac{\rho_{i+1} - \rho_i}{\rho_i + \kappa} \\
V(\rho_i) &= v_{\mathrm{f}}\cdot\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big] \\
y_1 &= q_i \\
y_2 &= v_i
\end{aligned} \tag{5}$$

In general, TS fuzzy systems (except for descriptor systems) do not contain algebraic constraints, and therefore q_i has to be eliminated. Also, consider q_{i-1}, v_{i-1}, and ρ_{i+1} inputs for the ith segment. Then, we have

$$\begin{aligned}
\dot\rho_i &= -\frac{1}{L}\rho_i v_i + \frac{1}{L\lambda}q_{i-1} \\
\dot v_i &= \frac{1}{\tau}(V(\rho_i) - v_i) - \frac{1}{L}v_i^2 \\
&\quad + \frac{\nu}{\tau\cdot L}\frac{\rho_i}{\rho_i+\kappa} \\
&\quad + \frac{1}{L}v_i v_{i-1} - \frac{\nu}{\tau\cdot L}\frac{\rho_{i+1}}{\rho_i+\kappa}
\end{aligned} \tag{6}$$

$$V(\rho_i) = v_{\mathrm{f}} \cdot \exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]$$

After some manipulations, we get

$$\begin{aligned}
\begin{pmatrix}\dot\rho_i\\ \dot v_i\end{pmatrix} &= \begin{pmatrix} -\frac{1}{L}v_i & 0\\ \frac{\nu}{\tau\cdot L}\frac{1}{\rho_i+\kappa} & -\frac{1}{\tau}-\frac{1}{L}v_i\end{pmatrix}\begin{pmatrix}\rho_i\\ v_i\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{\tau}V(\rho_i)\end{pmatrix}\\
&+\begin{pmatrix}\frac{1}{L\lambda} & 0 & 0\\ 0 & \frac{1}{L}v_i & -\frac{\nu}{\tau\cdot L}\frac{1}{\rho_i+\kappa}\end{pmatrix}\begin{pmatrix}q_{i-1}\\ v_{i-1}\\ \rho_{i+1}\end{pmatrix}\\
V(\rho_i) &= v_{\mathrm{f}}\cdot\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]
\end{aligned} \tag{7}$$

The variables q_{i-1}, v_{i-1}, and ρ_{i+1} depend on the states of the neighboring segments and $q_{i-1} = \rho_{i-1}v_{i-1}\lambda$, i.e., the product of two states of the previous segment. By rearranging the terms, we get

$$\begin{aligned}
\begin{pmatrix}\dot\rho_i\\ \dot v_i\end{pmatrix} &= \begin{pmatrix} 0 & -\frac{1}{L}\rho_i\\ \frac{\nu}{\tau\cdot L}\frac{1}{\rho_i+\kappa} & -\frac{1}{\tau}-\frac{1}{L}v_i\end{pmatrix}\begin{pmatrix}\rho_i\\ v_i\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{\tau}V(\rho_i)\end{pmatrix}\\
&+\begin{pmatrix}\frac{1}{L}\rho_{i-1} & 0\\ \frac{1}{L}v_i & -\frac{\nu}{\tau\cdot L}\frac{1}{\rho_i+\kappa}\end{pmatrix}\begin{pmatrix}v_{i-1}\\ \rho_{i+1}\end{pmatrix}\\
V(\rho_i) &= v_{\mathrm{f}}\cdot\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]
\end{aligned} \tag{8}$$

Note that this model is still fully equivalent to the original continuous-time model (5), and it is not an approximation. Starting from this model, we aim to obtain fuzzy representations of it.

B. Fuzzy modeling using sector nonlinearity

A method to obtain an exact fuzzy representation of a nonlinear dynamic system is by using the sector nonlinearity approach [13].

1) The approach: The main idea of obtaining a fuzzy model using the sector nonlinearity approach is as follows. For simplicity, consider a nonlinear system of the form

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{z})\boldsymbol{u}\\
\boldsymbol{y} &= C\boldsymbol{x}
\end{aligned} \tag{9}$$

with $\boldsymbol{f}$ and $\boldsymbol{g}$ smooth nonlinear matrix functions, $\boldsymbol{x} \in \mathcal{R}^n$ the state vector, $\boldsymbol{u} \in \mathcal{R}^{n_u}$ the input vector, and $\boldsymbol{y} \in \mathcal{R}^{n_y}$ the measurement vector, $\boldsymbol{z}$ some vector function of $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{u}$, all variables assumed to be bounded on a compact set $\mathcal{C}_{xyu}$.

Let $\mathrm{nl}_j(\cdot) \in [\underline{\mathrm{nl}}_j, \overline{\mathrm{nl}}_j]$, $j = 1, 2, \ldots, p$ be the set of bounded nonlinearities in $\boldsymbol{f}$ and $\boldsymbol{g}$, i.e., components of either $\boldsymbol{f}$ or $\boldsymbol{g}$. An exact TS fuzzy representation of (9) can be obtained by constructing first the weighting functions

$$w_0^j(\cdot) = \frac{\overline{\mathrm{nl}}_j - \mathrm{nl}_j(\cdot)}{\overline{\mathrm{nl}}_j - \underline{\mathrm{nl}}_j} \quad w_1^j(\cdot) = 1 - w_0^j(\cdot) \quad j = 1, 2, \ldots, p$$

and defining the membership functions as

$$h_i(\boldsymbol{z}) = \prod_{j=1}^{p} w_{i_j}^j(\boldsymbol{z}_j) \tag{10}$$

with $i = 1, 2, \cdots, 2^p$, $i_j \in \{0,1\}$. These membership functions are normal, i.e., $h_i(\boldsymbol{z}) \geq 0$ and $\sum_{i=1}^r h_i(\boldsymbol{z}) = 1$, $r = 2^p$, where r is the number of rules.

Using the membership functions defined in (10), an exact representation of (9) is given as:

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u})\\
\boldsymbol{y} &= C\boldsymbol{x}
\end{aligned} \tag{11}$$

with r the number of local linear models, A_i, B_i, $i = 1, 2, \ldots, r$ matrices of proper dimensions, and h_i, $i = 1, 2, \ldots, r$ defined as in (10).

2) The fuzzy flow model: Note that model (8) cannot be written[2] in the form (9). However, it can be written as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \boldsymbol{f}(\boldsymbol{z})\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{z})\boldsymbol{u} + \boldsymbol{a}(\boldsymbol{z})\\
\boldsymbol{y} &= \boldsymbol{h}(\boldsymbol{z})\boldsymbol{x}
\end{aligned} \tag{12}$$

i.e, with an affine term and a fuzzy measurement model.

In order to obtain a fuzzy representation of (8) similar to (11), consider first the state equations only.

A reasonable assumption is that ρ_{i-1}, ρ_i, and v_i are bounded, $\rho_{i-1} \in [\rho_{i-1,\min}, \rho_{i-1,\max}]$, $\rho_i \in [\rho_{i,\min}, \rho_{i,\max}]$, and $v_i \in [v_{i,\min}, v_{i,\max}]$. Since the minimum values can be taken nonnegative, the system (8) is well-defined on a compact set of variables. There are 5 nonlinearities, for which weighting functions are constructed, as follows:

1) $\frac{1}{L}\rho_i$ with the weighting functions $w_0^1 = \frac{\rho_{i,\max}-\rho_i}{\rho_{i,\max}-\rho_{i,\min}}$, $w_1^1 = 1 - w_0^1$;
2) $\frac{1}{\rho_i+\kappa}$ leads to $w_0^2 = \frac{\rho_i-\rho_{i,\min}}{\rho_i+\kappa}\frac{\rho_{i,\max}+\kappa}{\rho_{i,\max}-\rho_{i,\min}}$, $w_1^2 = 1 - w_0^2$;
3) $\frac{1}{\tau} + \frac{1}{L}v_i$, with $w_0^3 = \frac{v_{i,\max}-v_i}{v_{i,\max}-v_{i,\min}}$, $w_1^3 = 1 - w_0^3$; note that the nonlinearity $\frac{1}{L}v_i$ leads to the same weighting functions;
4) $\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]$, the nonlinearity in $V(\rho_i)$, with the weighting functions
$$w_0^4 = \frac{\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_{i,\min}}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big] - \exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_i}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]}{\exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_{i,\min}}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big] - \exp\Big[-\frac{1}{a_m}\Big(\frac{\rho_{i,\max}}{\rho_{\mathrm{cr}}}\Big)^{a_m}\Big]},$$
$w_1^4 = 1 - w_0^4$
5) $\frac{1}{L}\rho_{i-1}$ with the weighting functions $w_0^5 = \frac{\rho_{i-1,\max}-\rho_{i-1}}{\rho_{i-1,\max}-\rho_{i-1,\min}}$, $w_1^5 = 1 - w_0^5$;

Using these weighting functions a fuzzy model with $2^5 = 32$ rules is constructed.

Consider now the measurement function:

$$\boldsymbol{y} = \begin{pmatrix} q_i\\ v_i\end{pmatrix} = \begin{pmatrix} v_i\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho_i\\ v_i\end{pmatrix} \tag{13}$$

The nonlinearity $v_i\lambda$ in the measurement function leads to the weighting functions $w_0^6 = \frac{v_{i,\max}-v_i}{v_{i,\max}-v_{i,\min}}$, $w_1^6 = 1 - w_0^6$, i.e., the same weighting functions as in the above item 3. Since v_i is directly measured, the membership functions of the measurements do not depend on states to be estimated (the membership functions of the states depend on ρ_i, that is not measured).

In general, fuzzy representations of nonlinear systems obtained by the sector nonlinearity approach suffer from two

[2]To obtain the form (9), one has to divide by ρ_i, in which case the nonlinearities grow unbounded.

drawbacks. The first is that in general these local models do not necessarily retain the properties of the nonlinear system, e.g., for a globally observable nonlinear system one can obtain non-observable local models. However, for the obtained traffic model this is not the case, i.e., all local models are observable from the obtained measurement matrices.

A second drawback of the approach is the large number of rules obtained, in this case 32. To investigate whether a suitable fuzzy model with a smaller number of rules can be obtained, two other methods are tested next, that obtain approximations of the nonlinear system.

C. Fuzzy approximations of the traffic model

A first fuzzy approximation of the flow model was obtained by linearizing the nonlinear system in several operating points and interpolating the local models so obtained. Although by linearization in general a good approximation is obtained, in this case, due to the product of the variables, $3^5 = 243$ local models were necessary. Even for this number of local models, the norm of the difference of the true model and the approximation was as large as $6 \cdot 10^3$. For such a large number of rules, the observer design implies a large computational cost, actually exceeding the memory capacities of our machine. Therefore, the results obtained by linearization are not presented here.

Second, we used the method of [8], which is based on substitution instead of linearization. For this, we considered the model (8), with states ρ_i and v_i, and inputs v_{i-1}, ρ_{i+1}, ρ_{i-1}. For each of the variables ρ_i, v_i, and ρ_{i-1}, three operating points were chosen: the maximum, minimum, and average value. In each combination of these points, a local linear model has been obtained by substituting the chosen values in the matrices. With this method, 27 rules were obtained. However, this fuzzy model was an approximation of the model (8), with a bound on norm of the difference of the true model and the approximation as large as $2.5 \cdot 10^3$. This difference could have been reduced by adding more operating points, but the addition of even one more point leads to 36 rules, which is more than the number of rules obtained by the sector nonlinearity approach. Moreover, the model approximation error lead to a large bias in the estimation, and therefore the results are not presented here.

Due to the above presented drawbacks of the obtained approximate models, in the sequel we use the 32 rule TS model obtained by the sector nonlinearity approach.

IV. Observer design

A. Fuzzy observer design

Recall that the fuzzy model obtained in Section III-B.2 is of the form

$$\dot{\boldsymbol{x}} = \sum_{i=1}^{r} h_i(\boldsymbol{z})(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)$$
$$\boldsymbol{y} = \sum_{i=1}^{r} h_i(\boldsymbol{z})C_i\boldsymbol{x}$$

where the vector of scheduling variables $\boldsymbol{z}$ is a function of the state variables $\boldsymbol{x} = [\rho_i \; v_i]^T$, the input ρ_{i-1}, and the output v_i. Of these, ρ_i has to be estimated, and therefore, the fuzzy observer is of the following general form

$$\begin{aligned}\dot{\widehat{\boldsymbol{x}}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\widehat{\boldsymbol{x}} + B_i\boldsymbol{u} + a_i + K_i(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\\ \widehat{\boldsymbol{y}} &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})C_i\widehat{\boldsymbol{x}}\end{aligned} \tag{14}$$

where $\widehat{\boldsymbol{z}}$ denotes the estimated scheduling variables, and with the observer gains K_i, $i = 1, 2, \ldots, r$ computed such that the estimation error dynamics

$$\begin{aligned}\dot{\boldsymbol{e}} &= \dot{\boldsymbol{x}} - \dot{\widehat{\boldsymbol{x}}}\\ &= \sum_{i=1}^{r} h_i(\widehat{\boldsymbol{z}})(A_i\boldsymbol{e} - K_i(\boldsymbol{y} - \widehat{\boldsymbol{y}}))\\ &+ \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\end{aligned} \tag{15}$$

are asymptotically stable. Since v_i is measured, the membership functions of the measurement model do not depend on states to be estimated. Therefore, we have

$$\begin{aligned}\dot{\boldsymbol{e}} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\widehat{\boldsymbol{z}})h_j(\boldsymbol{z})(A_i - K_iC_j)\boldsymbol{e}\\ &+ \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\end{aligned} \tag{16}$$

Stability results for dynamics with such vanishing disturbances (i.e., $(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)$ disappears as $\widehat{\boldsymbol{z}}$ tends to $\boldsymbol{z}$) involve a Lipschitz condition on the disturbance: there exists a known $\mu > 0$ so that

$$\left\| \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\right\| \leq \mu\|\boldsymbol{e}\| \tag{17}$$

Under this condition, the dynamics (16) are asymptotically stable [2], if there exist P, K_i, $i = 1, 2, \ldots, r$, S so that

$$\begin{aligned}&\begin{pmatrix}\mathcal{H}(P(A_i - K_iC_i)) + \mu^2 S & P\\ P & -S\end{pmatrix} < 0\\ &i = 1, 2, \ldots, r\\ &\begin{pmatrix}\mathcal{H}(G_{ij}) + 2\mu^2 S & 2P\\ 2P & -2S\end{pmatrix} < 0\\ &i, j = 1, 2, \ldots, r \quad i \neq j\end{aligned} \tag{18}$$

with $G_{ij} = P(A_i - K_iC_j) + P(A_j - K_jC_i)$, where $\mathcal{H}(X)$ denotes the Hermitian of the matrix X, $\mathcal{H}(X) = X + X^T$.

B. Observer design for the fuzzy traffic model

The model obtained by the sector nonlinearity approach is the exact fuzzy representation of the model (5), and therefore the observer is actually designed for the model (5). It remains to be seen is whether (17) holds and whether (18) can be satisfied.

The membership functions h_i are the product of the weighting functions w_i. Since ρ_{i-1}, ρ_{i+1}, and v_{i-1} are assumed to be known from the neighboring segments, and v_i is measured, these values can be used in the weighting functions, i.e., one does not have to use the estimated value of v_i. Then, the three weighting functions that depend on the state ρ_i (that has to be estimated) are continuous on a compact set and therefore Lipschitz in $\boldsymbol{e}$. By analyzing these functions, we have:

$$\|w_0^1(\boldsymbol{z}) - w_0^1(\widehat{\boldsymbol{z}})\| \leq \|\boldsymbol{e}\| \frac{1}{\rho_{\max} - \rho_{\min}}$$
$$\|w_0^2(\boldsymbol{z}) - w_0^2(\widehat{\boldsymbol{z}})\| \leq \|\boldsymbol{e}\| \frac{\rho_{\max} + \kappa}{(\rho_{\max} - \rho_{\min})(\rho_{\min} + \kappa)}$$
$$\|w_0^4(\boldsymbol{z}) - w_0^4(\widehat{\boldsymbol{z}})\| \leq \|\boldsymbol{e}\| \cdot \frac{1}{\exp\left[-\frac{1}{a_m}\left(\frac{\rho_{i,\min}}{\rho_{\text{cr}}}\right)^{a_m}\right] - \exp\left[-\frac{1}{a_m}\left(\frac{\rho_{i,\max}}{\rho_{\text{cr}}}\right)^{a_m}\right]}$$

The same bounds hold for w_1^i, $i = 1, 2, 4$. Moreover, for each membership function we have

$$h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}) = \prod_{j=1}^{5} w_{i_j}^j(\boldsymbol{z}) - \prod_{j=1}^{5} w_{i_j}^j(\widehat{\boldsymbol{z}}) \leq \sum_{j=1}^{5}(w_{i_j}^j(\boldsymbol{z}) - w_{i_j}^j(\widehat{\boldsymbol{z}}))$$

with $i_j \in \{0, 1\}$. Therefore, $h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}})$ is Lipschitz in $\boldsymbol{z} - \widehat{\boldsymbol{z}}$, with the Lipschitz constant smaller than or equal to the number of rules times the sum of the Lipschitz constants of the weighting functions. For the values in Table II, an upper bound can be obtained as $\|h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}})\| \leq 0.1608\|\boldsymbol{e}\|$. Moreover, since all variables are defined on a compact set, $\|A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i\|$ is bounded, in this case $\|A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i\| \leq 1.22 \cdot 10^5$.

Using this bound, (18) can be written as an LMI problem, and solved using Matlab's *feasp* function. Note however, that this bound represents the worst-case scenario and is rarely needed. In fact, as long as the estimated initial states are close enough to the real ones, if (18) has a solution, the estimated states will converge to the true ones. Therefore, for solving (18), we considered $\mu = 100$ and computed the matrices P, K_i, $i = 1, 2, \ldots, r$, and S. For randomly generated ρ_{i-1}, ρ_{i+1}, and v_{i-1}, true initial states $\boldsymbol{x}_0 = [15, 120]^T$, and estimated initial states $\widehat{\boldsymbol{x}}_0 = [10, 70]^T$, the estimation error trajectory is presented in Figure 2. As expected, the estimation error converges to zero, i.e., the estimated states converge to the true states. For the numerical integration the Matlab function *ode45* was used.

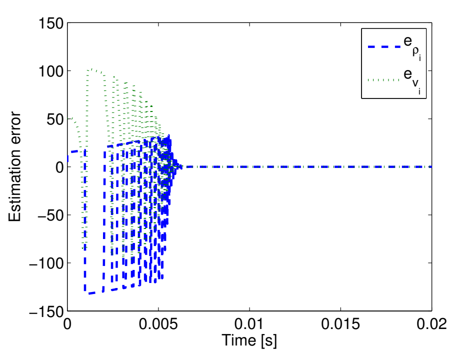

Fig. 2. Estimation error.

C. Robust observer

Recall that in the traffic flow model (2), the average velocity is in fact a heuristic relationship, and a noise is considered to account for the model mismatch. Also, the measurements are in general not accurate, i.e., the correct measurement model is (4). In order to attenuate the effect of these disturbances on the estimated states, we consider a robust fuzzy observer.

A fuzzy system of the form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \sum_{i=1}^{r} h_i(\boldsymbol{z})(E_i\boldsymbol{x} + F_i\epsilon) \\ \boldsymbol{y} &= G\boldsymbol{x} \end{aligned} \tag{19}$$

is asymptotically stable [6] with γ disturbance attenuation (i.e., the effect of the disturbance on the output is attenuated by a factor γ) under zero initial conditions ($\boldsymbol{x}(0) = 0$), if there exists $P = P^T > 0$ such that

$$\begin{pmatrix} \mathcal{H}(PE_i) + G^TG & PF_i \\ F_i^TP & -\gamma^2 I \end{pmatrix} < 0, \quad i = 1, 2, \ldots, r \tag{20}$$

holds.

Taking into account the corrupting noises, and the observer designed in Section III-B.2, the estimation error dynamics can be written as

$$\begin{aligned} \dot{\boldsymbol{e}} &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\widehat{\boldsymbol{z}})h_j(\boldsymbol{z})\Big[(A_i - K_iC_j)\boldsymbol{e} - K_i\begin{pmatrix}\zeta_q\\ \zeta_v\end{pmatrix}\Big] \\ &\quad + \begin{pmatrix}0\\1\end{pmatrix}\xi_v + \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\widehat{\boldsymbol{z}})h_j(\boldsymbol{z})\Big[(A_i - K_iC_j)\boldsymbol{e} \\ &\quad + \begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} & -K_i\end{pmatrix}\begin{pmatrix}\xi_v\\ \zeta_q\\ \zeta_v\end{pmatrix}\Big] \\ &\quad + \sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i) \end{aligned} \tag{21}$$

Combining the conditions (20) and (18), we have that the dynamics (21) is asymptotically stable with γ disturbance attenuation, under zero initial condition, if there exist μ, P, K_i, $i = 1, 2, \ldots, r$, S so that

$$\begin{gathered} \Big\|\sum_{i=1}^{r}(h_i(\boldsymbol{z}) - h_i(\widehat{\boldsymbol{z}}))(A_i\boldsymbol{x} + B_i\boldsymbol{u} + a_i)\Big\| \leq \mu\|\boldsymbol{e}\| \\ \begin{pmatrix} \mathcal{H}(G_i) + \mu^2 S + I & P & P\left(\begin{pmatrix}0\\1\end{pmatrix} - K_i\right) \\ P & -S & 0 \\ \left(\begin{pmatrix}0\\1\end{pmatrix} - K_i\right)^T P & 0 & -\gamma^2 I \end{pmatrix} < 0 \\ \begin{pmatrix} \mathcal{H}(G_{ij}) + 2\mu^2 S + 2I & 2P & 2P\left(\begin{pmatrix}0\\1\end{pmatrix} - K_i\right) \\ 2P & -2S & 0 \\ 2P\left(\begin{pmatrix}0\\1\end{pmatrix} - K_i\right)^T P & 0 & -\gamma^2 I \end{pmatrix} < 0 \end{gathered} \tag{22}$$

where $G_i = P(A_i - K_iC_i)$ $G_{ij} = P(A_i - K_iC_j) + P(A_j - K_jC_i)$.

The above conditions were written in an LMI form, and solved with Matlab's *mincx*, that minimizes a linear objective under LMI conditions. With $\mu = 100$, as in the previous case, the maximum disturbance attenuation that can be achieved is $\gamma = 0.08$. The estimation error under zero initial conditions, for a randomly generated trajectory can be seen in Figure 3. For this trajectory, the noise signals were generated according to a uniform distribution, $\xi_v \in [-5, 5]$, $\zeta_q \in [-10, 10]$, and $\zeta_v \in [-5, 5]$, and for the numerical integration the Matlab function *ode45* was used. As can be seen, the effect of the disturbance on the estimation error was attenuated with even more than the computed factor γ.

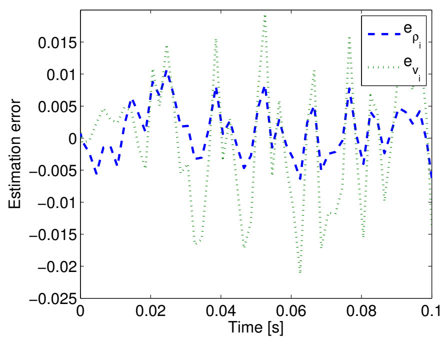

Fig. 3. Estimation error with γ disturbance attenuation under zero initial conditions.

V. Conclusions

In this paper we have proposed a fuzzy observer for the well-known second-order traffic flow model METANET in order to estimate the non-measured states. To design the observer, first a dynamic Takagi-Sugeno fuzzy model was derived using the sector nonlinearity approach. This fuzzy model is an exact representation of the continuous-time traffic flow model. For the obtained fuzzy model, fuzzy observers were designed, and also disturbance attenuation has been achieved.

In this paper the modeling and observer design was performed in continuous time. Since the METANET model has been validated in discrete time, in our future research, we will derive a discrete-time observer, that can also be tested on measured data. A drawback of the model developed is that the number of rules grows exponentially with the number of segments considered. Therefore, we will investigate distributed observer design.

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