An alternative LMI static output feedback control design for discrete-time nonlinear systems represented by Takagi-Sugeno models

***Abstract*—This paper presents a static output feedback controller design for discrete-time nonlinear systems exactly represented by Takagi-Sugeno models. By introducing past states in the control law as well as in the Lyapunov function, more relaxed results are obtained. Different conditions in terms of linear matrix inequalities are provided, whose structure depends on the well-known Finsler's Lemma. The proposed conditions are less demanding than the ones in the literature. This is illustrated via numerical examples.**

***Index Terms*—Static output feedback; nonlinear systems; delayed non-quadratic Lyapunov function; linear matrix inequalities.**

I. INTRODUCTION

Takagi-Sugeno (TS) [1] models have gained reputation as an important tool for the analysis and control of nonlinear systems. Via the sector nonlinearity methodology [2] a nonlinear model can be exactly represented by a TS one. A TS model is a collection of local models blended together by scalar membership functions (MFs). Thanks to this convex structure, the direct Lyapunov method can be applied [3], [4]. The aim is to cast conditions in terms of linear matrix inequalities (LMIs), which can be solved via convex optimization techniques [5]. Nonetheless, within the TS-LMI framework, the derived LMI conditions are only sufficient and may be conservative. Sources of conservativeness are: the type of Lyapunov function, the non-uniqueness of the TS model, the way the MFs are dropped off from the inequality expressions, etc.

Since the appearance of the Parallel Distributed Compensation (PDC) technique [6] together with quadratic Lyapunov functions, the design of state feedback controllers has been widely studied: non-quadratic Lyapunov functions (fuzzy ones) [7]–[10], piecewise Fuzzy Lyapunov functions [11]–[13], asymptotically necessary and sufficient conditions [14], [15], delayed non-quadratic Lyapunov functions together with delayed non-PDC controllers [16], [17] have been employed. A pole-placement-like technique for TS models has been introduced in [18] while several polynomial approaches are gathered in the book [19].

When full information of the state is not accessible, one alternative is the use of state observers [20]; for the observer design several LMI approaches are available [21]–[23]. Another alternative is the use of output feedback controllers [24]; within the TS-LMI framework some works concern the Static Output Feedback Controller (SOFC) design problem [25]–[29]. Such an approach leads to bilinear matrix inequalities (BMIs), which are not efficiently solved via convex optimization techniques. Several attempts to translate the BMI into an LMI problem has been done. For instance, authors in [29] have developed iterative LMI (ILMI) conditions for designing a SOFC, similar to those in [30]. By extending the results of the linear case [31], sufficient LMI constraints have been achieved in [27]. Later, using the so-called descriptor redundancy, an LMI solution has been provided in [25]. Recently, in the discrete-time case, an LMI solution has been given in [26], where the authors have employed the descriptor redundancy together with Finsler's lemma. The use of such tools avoids the existence of undesirable 'crossed' products between the control gains and the Lyapunov matrix, thus an LMI formulation can be achieved. Nevertheless, the controller proposed therein is a PDC-like SOFC and it has not exploited the discrete-time nature of the problem.

Summarizing, within the TS context, several approaches for SOFC exists, nevertheless obtaining LMI conditions has been done through conservative Lyapunov functions, complex LMI conditions with many decision variables

The contribution of this paper is threefold: 1) a straightforward relaxation of the conditions given in [26] by using delayed non-PDC controllers; 2) to provide an alternative to the SOFC design for discrete-time TS models by giving simpler LMI conditions than [26]; 3) a unification of both methodologies. This unification consists in developing LMI conditions that include the feasible solution sets of both approaches.

The paper is organized as follows: Section II presents the TS models, lemmas and properties; Section III states the problem to be solved and motives this research; Section IV establishes the main results of our work and illustrates them via numerical examples; Section V concludes the paper with some final remarks and discussions.

II. PRELIMINARIES

A. From nonlinear models to Takagi-Sugeno ones

Consider a discrete-time affine-in-the-control nonlinear model:

$$\begin{aligned} x_{k+1} &= f(x_k)x_k + g(x_k)u_k \\ y_k &= s(x_k)x_k, \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state, $u_k \in \mathbb{R}^{n_u}$ is the input, $y_k \in \mathbb{R}^{n_y}$ is the output vector, and k is the current sample. Matrices $f(\cdot)$, $g(\cdot)$, and $s(\cdot)$ are assumed to be bounded and smooth in a compact set Ω_x of the state space.

The methodology to express (1) as a convex model is called the sector nonlinearity approach [2]. It begins by identifying nonlinear terms[1] $z_1(x_k), z_2(x_k), \ldots, z_p(x_k)$ in (1). Knowing their bounds the compact set Ω_x, i.e., $z_j(x_k) \in [\underline{z}_j \quad \overline{z}_j]$, $j \in \{1,2,\ldots,p\}$, these terms can be rewritten as convex sums:

$$z_j(x_k) = w_0^j \overline{z}_j + w_1^j \underline{z}_j, \tag{2}$$

where $w_0^j = \left(z_j(x_k) - \underline{z}_j\right) / \left(\overline{z}_j - \underline{z}_j\right)$ and $w_1^j = 1 - w_0^j$ are weighting functions (WFs). The WFs hold the convex sum property $w_0^j + w_1^j = 1$, $w_i^j \in [0,1]$ in Ω_x. Thanks to the convexity of the terms $z_1(x_k), z_2(x_k), \ldots, z_p(x_k)$, the nonlinear model (1) can be *exactly* rewritten in Ω_x as

$$\begin{aligned} x_{k+1} &= \sum_{i=1}^{r} h_i(z(x_k))(A_i x_k + B_i u_k) \\ y_k &= \sum_{i=1}^{r} h_i(z(x_k)) C_i x_k, \end{aligned} \tag{3}$$

where $h_i(z(x_k)) = \prod_{j=1}^{p} w_{i_j}^j(z_j(x_k))$, $i \in \{1,2,\ldots,2^p\}$, $i_j \in \{0,1\}$ are called membership functions (MFs),

[1] In this work, these nonlinear terms depend exclusively on measurable states; the unmeasurable states cannot be part of the MFs, the latter case is left out of the current research. Many works have been done in order to tackle this drawback, for instance see [22], [23], [32]–[34].

$r=2^p$ is the number of vertices in (3). Each tuple (A_i, B_i, C_i) is defined at the vertex $h_i=1$. By construction, the MFs hold the convex sum property in Ω_x: $\sum_{i=1}^{r} h_i(z(x_k))=1$, $h_i(z_k(x_k))\in[0,1]$, $i\in\{1,2,\ldots,r\}$. Arguments will be omitted when their meaning can be inferred from the context.

B. Properties and lemmas

In order to obtain LMI conditions, MFs are usually dropped out from the expression; to this end the following sum relaxation scheme will be employed.

Lemma 1. [35] (Relaxation Lemma). Let $\Upsilon_{ij}^{l}=(\Upsilon_{ij}^{l})^{T}$, $(i,j,l)\in\{1,2,\ldots,r\}^{3}$ be matrices of adequate dimensions. If

$$\frac{2}{r-1}\Upsilon_{ii}^{l}+\Upsilon_{ij}^{l}+\Upsilon_{ji}^{l}<0,\quad \forall i,j,l \tag{4}$$

then $\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{l=1}^{r} h_i(z(x_k))h_j(z(x_k))h_l(z(x_k))\Upsilon_{ij}^{l}<0$ holds, is true as long as the MFs hold the convex sum property $h_i(z(x_k))\in[0,1]$, $\sum_{i=1}^{r} h_i(z(x_k))=1$, $i\in\{1,2,\ldots,r\}$.

In further developments the following lemma is used.

Lemma 2. [36] (Finsler's lemma). Let $\mathrm{X}\in\mathbb{R}^{n}$, $\mathrm{Q}=\mathrm{Q}^{T}\in\mathbb{R}^{n\times n}$, and $\mathrm{W}\in\mathbb{R}^{m\times n}$ such that $rank(\mathrm{W})<n$; the following expressions are equivalent:

a) $\mathrm{X}^{T}\mathrm{Q}\mathrm{X}<0,\quad \forall \mathrm{X}\in\{\mathrm{X}\in\mathbb{R}^{n}:\mathrm{X}\neq 0, \mathrm{W}\mathrm{X}=0\}$.
b) $\exists \mathrm{M}\in\mathbb{R}^{n\times m}:\mathrm{Q}+\mathrm{M}\,\mathrm{W}+\mathrm{W}^{T}\mathrm{M}^{T}<0$.

C. Notation

Throughout this paper, the following shorthand notation is adopted to conveniently represent convex sums of matrix expressions such as: $\Upsilon_h=\sum_{i=1}^{r} h_i(z(x_k))\Upsilon_i$ and its inverse $\Upsilon_h^{-1}=\left(\sum_{i=1}^{r} h_i(z(x_k))\Upsilon_i\right)^{-1}$; with delayed MFs $\Upsilon_{h^+}=\sum_{l=1}^{r} h_i(z(x_{k+1}))\Upsilon_l$, $\Upsilon_{h^-}=\sum_{l=1}^{r} h_i(z(x_{k-1}))\Upsilon_l$; or multiple convex sums

$\Upsilon_{hh}=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i\left(z\left(x_k\right)\right)h_j\left(z\left(x_k\right)\right)\Upsilon_{ij}$. Using the aforementioned notation, the TS model (3) is shortly written as:

$$x_{k+1}=A_h x_k+B_h u_k, \qquad y_k=C_h x_k.$$

An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side, for example:

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}=\begin{bmatrix} A & (*) \\ B & C \end{bmatrix}, \; A+B+A^T+B^T+C=A+B+(*)+C.$$

D. Problem statement

The goal is to design a static output feedback controller (SOFC) for the TS model (3). For instance, in [26], the following PDC control has been proposed

$$u_k=\underbrace{\sum_{j=1}^{r}h_j\left(z\left(x_k\right)\right)K_j}_{K_h}y_k. \tag{5}$$

The classical closed-loop model writes:

$$x_{k+1}=\left(A_h+B_h K_h C_h\right)x_k, \tag{6}$$

from which it is difficult to get a pure LMI constraint problem [27]. Some works have tried to overcome this, for example, conditions in [27] are given as a set of LMIs *together* with equality constraints, which for different output matrices lead to a set of equality constraints very hard to be satisfied. Another way to tackle this problem has been provided in [26] applying both the descriptor-redundancy and Finsler's lemma. Effectively, by using the so-called descriptor redundancy, the TS model (3) and the control law (5) are expressed as:

$$\bar{E}\bar{x}_{k+1}=\bar{A}_{hh}\bar{x}_k, \tag{7}$$

with $\bar{x}_k=\begin{bmatrix} x_k \\ u_k \end{bmatrix}$, $\bar{E}=\begin{bmatrix} I_{n_x} & 0 \\ 0 & 0_{n_u} \end{bmatrix}$, $\bar{A}_{hh}=\begin{bmatrix} A_h & B_h \\ K_h C_h & -I_{n_u} \end{bmatrix}$. At last, (7) is written as an equality constraint:

$$\begin{bmatrix} \bar{A}_{hh} & -I_{n_x+n_u} \end{bmatrix} \begin{bmatrix} \bar{x}_k \\ \bar{E}\bar{x}_{k+1} \end{bmatrix} = 0. \tag{8}$$

Consider the following Lyapunov function candidate

$$V(\bar{x}_k) = \bar{x}_k^T \bar{E}^T \bar{P}_h \bar{E}\bar{x}_k, \ \bar{P}_h = \begin{bmatrix} P_{1h} & P_{2h} \\ P_{2h}^T & P_{3h} \end{bmatrix}, \tag{9}$$

$P_{1h} > 0 \in \mathbb{R}^{n_x \times n_x}$, $P_{3h} = P_{3h}^T \in \mathbb{R}^{n_u \times n_u}$. Via Lemma 2, the variation of (9) subject to constraint (8) can be expressed as

$$\mathrm{M} \begin{bmatrix} \bar{A}_{hh} & -I_{n_x+n_u} \end{bmatrix} + (*) + \begin{bmatrix} -\bar{E}^T \bar{P}_h \bar{E} & 0 \\ 0 & \bar{P}_{h^+} \end{bmatrix} < 0, \tag{10}$$

with $\mathrm{M} \in \mathbb{R}^{2(n_x+n_u)\times(n_x+n_u)}$ is matrix to be defined later. Thus, the following result has been stated:

Lemma 3 [26]. The nonlinear model (1) under the control law (5) has the origin asymptotically stable if there exist matrices $P_{1j} = P_{1j}^T > 0$, P_{2j}, $P_{3j} = P_{3j}^T$, G, G_{1jk}, G_{2jk}, H_{1jk}, H_{2jk}, and N_j such that (4) holds with:

$$\Upsilon_{ij}^l = \begin{bmatrix} \Gamma^{(1,1)} & (*) & (*) & (*) \\ \Gamma^{(2,1)} & \Gamma^{(2,2)} & (*) & (*) \\ \Gamma^{(3,1)} & \Gamma^{(3,2)} & -H_{1jl} - (H_{1jl})^T + P_{1l} & (*) \\ \Gamma^{(4,1)} & \Gamma^{(4,2)} & -H_{2jl} - (\mathrm{J}\, G)^T + P_{2l} & -G - G^T + P_{3l} \end{bmatrix}, (i, j, l) \in \{1, 2, \ldots, r\}^3;$$

where

$$\Gamma^{(1,1)} = G_{1jl}A_i + \mathrm{J}\, N_j C_i + (*) - P_{1j},$$
$$\Gamma^{(2,1)} = G_{2jl}A_i + N_j C_i + (G_{1jl}B_i - \mathrm{J}\, G)^T,$$
$$\Gamma^{(3,1)} = H_{1jl}A_i + \mathrm{J}\, N_j C_i - G_{1jl}^T,$$
$$\Gamma^{(4,1)} = H_{2jl}A_i + N_j C_i - (\mathrm{J}\, G)^T, \Gamma^{(2,2)} = G_{2jl}B_i - G + (*),$$
$$\Gamma^{(3,2)} = H_{1jl}B_i - \mathrm{J}\, G - G_{2jl}^T, \Gamma^{(4,2)} = H_{2jl}B_i - G - G^T.$$

The controller gains are computed by $K_j = G^{-1}N_j$, $j \in \{1, 2, \ldots, r\}$.

Proof: Take (10) and chose $\mathrm{M} = \begin{bmatrix} G_{1hh^+}^T & G_{2hh^+}^T & H_{1hh^+}^T & H_{2hh^+}^T \\ G^T \mathcal{J}^T & G^T & G^T \mathcal{J}^T & G^T \end{bmatrix}^T$, it directly yields the desired result. ■

Remark 1. The results in Lemma 3 are LMIs up to fixing a priori the matrix $\mathcal{J} \in \mathbb{R}^{n_x \times n_u}$. This matrix can be chosen as $\mathcal{J} = 0_{n_x \times n_u}$, $\mathcal{J} = B_h$, etc. Different configurations lead to different LMI problems, whose solution set may differ or overlap, i.e., their conservatism depends on the problem under study [26].

Remark 2. Within the LMI context, several closed-loop performances can be directly added. For instance, input/output constraints, convergence speed (exponential stability) or a general approach like D-stability, disturbance attenuation via H∞ [5], [21], [37].

Note that methodology given in [26] first rewrites the TS model (3) together with the control law (5) by means of the so-called descriptor-redundancy forcing a singular system structure. Then well-known Finsler's lemma is used in order to conveniently introduce slack variables into the final conditions.

The results of Lemma 3 can be significantly outperformed using delayed-non-PDC control laws associated with delayed Lyapunov functions, inspired by the recent results of [17], i.e. using:

$$u = \left(G_{hhh^-}\right)^{-1} K_{hh^-} y, \tag{11}$$

where $K_{hh^-} = \sum_{j=1}^{r} \sum_{l=1}^{r} h_j\left(z\left(x_k\right)\right) h_l\left(z\left(x_{k-1}\right)\right) K_{jl}$ and $G_{hhh^-} = \sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{l=1}^{r} h_i\left(z\left(x_k\right)\right) h_j\left(z\left(x_k\right)\right) h_l\left(z\left(x_{k-1}\right)\right) G_{ijl}$.

Moreover, (11) allows achieving relaxed results *without* increasing the number of LMIs, but naturally adding more decision variables [8], [17]. The latter result is summarized in the following theorem.

Theorem 1. The nonlinear model (1) under the control law (11) has the origin asymptotically stable if there exist matrices $P_{1j} = P_{1j}^T > 0$, P_{2j}, $P_{3j} = P_{3j}^T$, G_{ijl}, G_{1jl}, G_{2jl}, H_{1jl}, H_{2jl}, and K_{jl} such that (4) holds with:

$$\Upsilon_{ij}^{l} = \begin{bmatrix} \Gamma^{(1,1)} & (*) & (*) & (*) \\ \Gamma^{(2,1)} & \Gamma^{(2,2)} & (*) & (*) \\ \Gamma^{(3,1)} & \Gamma^{(3,2)} & -H_{1jl} - H_{1jl}^T + P_{1j} & (*) \\ \Gamma^{(4,1)} & \Gamma^{(4,2)} & -H_{2jl} - \left(\mathcal{J}\, G_{ijl}\right)^T + P_{2j} & -G_{ijl} - G_{ijl}^T + P_{3j} \end{bmatrix}, \quad (i, j, l) \in \{1, 2, \ldots, r\}^3$$

where

$$\Gamma^{(1,1)} = G_{1jl}A_i + \mathrm{J}\, K_{jl}C_i + (*) - P_{1l},$$
$$\Gamma^{(2,1)} = G_{2jl}A_i + K_{jl}C_i + \left(G_{1jl}B_i - \mathrm{J}\, G_{ijl}\right)^T,$$
$$\Gamma^{(3,1)} = H_{1jl}A_i + \mathrm{J}\, K_{jl}C_i - G_{1jl}^T,$$
$$\Gamma^{(4,1)} = H_{2jl}A_i + K_{jl}C_i - \left(\mathrm{J}\, G_{ijl}\right)^T,$$
$$\Gamma^{(2,2)} = G_{2jl}B_i - G_{ijl} + (*),$$
$$\Gamma^{(3,2)} = H_{1jl}B_i - \mathrm{J}\, G_{ijl} - G_{2jl}^T,$$
$$\Gamma^{(4,2)} = H_{2jl}B_i - G_{ijl} - G_{ijl}^T.$$

Proof: It follows similar developments as Lemma 3, but by considering the control law (11) and the following delayed Lyapunov function candidate

$$V(\bar{x}) = \bar{x}^T \bar{E}^T \bar{P}_{h^-} \bar{E}\bar{x},\ \ \bar{P}_{h^-} = \begin{bmatrix} P_{1h^-} & P_{2h^-} \\ P_{2h^-}^T & P_{3h^-} \end{bmatrix}, P_{1h^-} > 0, P_{3h^-} = P_{3h^-}^T. \tag{12}$$

where $P_{ih^-} = \sum_{l=1}^{r} h_l\left(z\left(x_{k-1}\right)\right) P_{il}$, $i = 1,2,3$. For this case, the Finsler matrix M is chosen accordingly

$$\mathrm{M} = \begin{bmatrix} G_{1hh^-}^T & G_{2hh^-}^T & H_{1hh^-}^T & H_{2hh^-}^T \\ G_{hhh^-}^T & G_{hhh^-}^T & G_{hhh^-}^T \mathrm{J}^T & G_{hhh^-}^T \end{bmatrix}^T. \ \blacksquare$$

The validity of the control law (11) and the delayed Lyapunov function (12) has been discussed in [16], [17]. Despite the fact that Theorem 1 incorporates recent advances in the TS-LMI framework, limitations still exist. The following example motives the rest of the study in this paper: it uses previous approach in Lemma 3 [26] and its direct improvement in Theorem 1; both approaches are found *unfeasible*.

Example 1. Consider a nonlinear system (1) with $f(x_k) = \begin{bmatrix} 0.2 + 0.12\cos x_1 & 1.6 \\ -0.8 & 0 \end{bmatrix}$, $g(x_k) = \begin{bmatrix} 0.1 \\ -2 - 1.04\sin x_1 \end{bmatrix}$, and $s(x_k) = \begin{bmatrix} 0.2 + 0.1\cos x_1 & 0 \end{bmatrix}$. Using the sector nonlinearity approach, with $z_1 = \cos x_1 \in [-1,1]$ and $z_2 = \sin x_1 \in [-1,1]$, a TS model of the form (3) is obtained, with vertex matrices as follows: $A_1 = A_2 = \begin{bmatrix} 0.32 & 1.6 \\ -0.8 & 0 \end{bmatrix}$, $A_3 = A_4 = \begin{bmatrix} 0.08 & 1.6 \\ -0.8 & 0 \end{bmatrix}$, $B_1 = B_3 = \begin{bmatrix} 0.1 \\ -3.04 \end{bmatrix}$, $B_2 = B_4 = \begin{bmatrix} 0.1 \\ -0.96 \end{bmatrix}$,

$C_1 = C_2 = \begin{bmatrix} 0.3 & 0 \end{bmatrix}$, and $C_3 = C_4 = \begin{bmatrix} 0.1 & 0 \end{bmatrix}$. Only the state x_1 is available for control purposes. The MFs are $h_1(x_1) = w_0^1 w_0^2$, $h_2(x_1) = w_0^1 w_1^2$, $h_3(x_1) = w_1^1 w_0^2$, and $h_4(x_1) = w_1^1 w_1^2$, where the weighting functions are $w_0^1 = 0.5(\cos x_1 + 1)$, $w_0^2 = 0.5(\sin x_1 + 1)$, $w_1^1 = 1 - w_0^1$, and $w_1^2 = 1 - w_0^2$. For this example neither conditions in Lemma 3, under Remark 1, nor the ones in Theorem 1 provide feasible solutions. ♦

Thus, the goal of this paper is to provide an alternative for the SOFC design by the use of different configurations of the control and at the same time, to reduce the computational burden of the LMI conditions.

III. Main results

The results in this section are based on Lemma 2. Let us first consider the PDC-like SOFC (5) together with the TS model (3)

$$\underbrace{\begin{bmatrix} A_h & -I_{n_x} & B_h \\ K_h C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix}}_{\mathrm{W}} \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}}_{\mathrm{X}} = 0. \qquad (13)$$

which already avoids writing the closed-loop (6). Consider also the following Lyapunov function candidate $\Delta V(x) = x_k^T P_h x_k > 0$ with $P_h > 0$, and $P_h = \sum_{j=1}^{r} h_j(z(x_k)) P_j$; whose variation is given by $\Delta V(x) = x_{k+1}^T P_{h^+} x_{k+1} - x_k^T P_h x_k$. Note that $\Delta V(x)$ can be arranged as:

$$\Delta V = \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}^T}_{\mathrm{X}} \underbrace{\begin{bmatrix} -P_h & 0 & 0 \\ 0 & P_{h^+} & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix}}_{\mathrm{Q}} \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}}_{\mathrm{X}}. \qquad (14)$$

Then, by means of Lemma 2, $\Delta V < 0$ holds for all $\mathrm{X} \neq 0$ under the constraint (13) if there exist $\mathrm{M} \in \mathbb{R}^{(2n_x + n_u) \times (n_x + n_u)}$ such that

$$\mathrm{M}\begin{bmatrix} A_h & -I_{n_x} & B_h \\ K_h C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix} + (*) + \begin{bmatrix} -P_h & 0 & 0 \\ 0 & P_{h^+} & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix} < 0, \tag{15}$$

where M is a free matrix to be chosen a priori. Its structure will be discussed for each case. Hence, the following result can be stated.

Theorem 2. The nonlinear model (1) under the control law (5) has the origin asymptotically stable if there exist matrices $P_j = P_j^T > 0$, G, H_{jk}, and N_j such that (4) holds with:

$$\Upsilon_{ij}^{l} = \begin{bmatrix} -P_j & (*) & (*) \\ H_{jl}A_i + \mathrm{J}\, N_j C_i & -H_{jl} - H_{jl}^T + P_l & (*) \\ N_j C_i & \left(H_{jl}B_i - \mathrm{J}\, G\right)^T & -G - G^T \end{bmatrix}, \ (i, j, l) \in \{1, 2, \ldots, r\}^3. \tag{16}$$

The control gains are computed as $K_j = G^{-1}N_j$.

Proof. Recall (15). Choose

$$\mathrm{M} = \begin{bmatrix} 0_{n_x} & 0_{n_x \times n_u} \\ H_{hh^+} & \mathrm{J}\, G \\ 0_{n_u \times n_x} & G \end{bmatrix}, \ H_{hh^+} \in \mathbb{R}^{n_x \times n_x}, \quad G \in \mathbb{R}^{n_u \times n_u}. \tag{17}$$

Equation (15) yields

$$\begin{bmatrix} -P_h & (*) & (*) \\ H_{hh^+}A_h + \mathrm{J}\, N_h C_h & -H_{hh^+} - H_{hh^+}^T + P_{h^+} & (*) \\ N_h C_h & \left(H_{hh^+}B_h - \mathrm{J}\, G\right)^T & -G + G^T \end{bmatrix} < 0,$$

where $N_h = GK_h$; finally, by means of Lemma 1 the proof is concluded. ■

Let us test the results of Theorem 2, by setting $\mathrm{J} = B_h$ and resuming the previous example where no solution was found for the approach in [26].

Example 1 (continued). Using the conditions in Theorem 2, with the PDC control law (5), the following values have been obtained:

$$P_1 = P_2 = \begin{bmatrix} 0.1278 & 0.0266 \\ 0.0266 & 0.3835 \end{bmatrix}, P_3 = P_4 = \begin{bmatrix} 0.1345 & 0.0041 \\ 0.0041 & 0.3936 \end{bmatrix}, G = 0.3998, K_1 = K_2 = -0.1926, K_3 = K_4 = -2.5005.$$

Once the controller gains are computed, the simulations are conducted using the nonlinear system, that is $x_{k+1} = \left(f\left(x_k\right) + g\left(x_k\right) K_h \right) x_k$. with $K_h = \sum_{j=1}^{4} h_j\left(z\left(x_k\right)\right) K_j$; the state trajectories are displayed in Figure 1 for initial conditions $x(0) = \begin{bmatrix} 0.9 & -0.5 \end{bmatrix}^T$; the state x_1 is represented by a black-solid-line while x_2 is in blue-dashed-line. It can be seen that the open-loop system exhibits an unstable behavior, while in Figure 1 (b), corresponding to the closed-loop, the designed SOFC drives the states to zero.

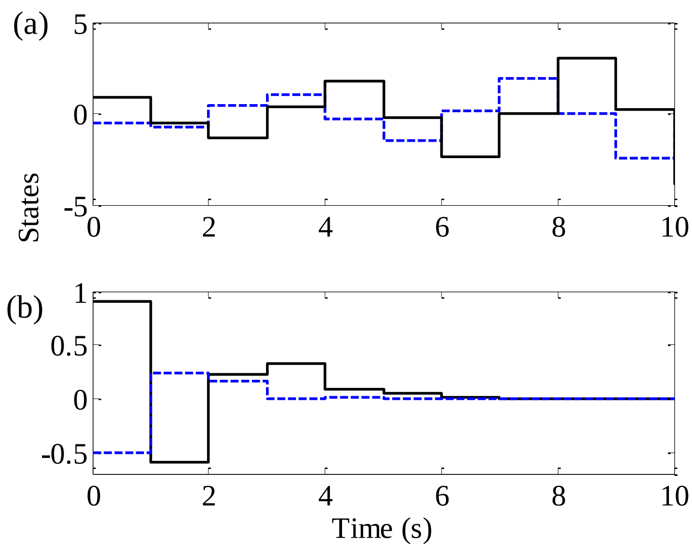

Figure 1. (a) State trajectories of the open-loop system. (b) State trajectories of the closed-loop system. ♦

The following result is a direct improvement of conditions in Theorem 2, it employs the delayed approach in [17].

Theorem 3: The nonlinear model (1) under the control law (11) has the origin asymptotically stable if there exist matrices $P_j = P_j^T > 0$, G_{ijl}, H_{jl}, and K_{jl} such that (4) holds with:

$$\Upsilon_{ij}^{l} = \begin{bmatrix} -P_l & (*) & (*) \\ H_{jl} A_i + \mathrm{J}\, K_{jl} C_i & -H_{jl} - H_{jl}^T + P_j & (*) \\ K_{jl} C_i & \left(H_{jl} B_i - \mathrm{J}\, G_{ijl}\right)^T & -G_{ijl} - G_{ijl}^T \end{bmatrix}. \qquad (18)$$

Proof. Consider a delayed Lyapunov function of the form $V = x_k^T P_{h^-} x_k$ with $P_{h^-} > 0$, and $P_{h^-} = \sum_{l=1}^{r} h_l\left(z\left(x_{k-1}\right)\right) P_l$; the variation $\Delta V = x_{k+1}^T P_h x_{k+1} - x_k^T P_{h^-} x_k$ together with the system dynamics (3) under the delayed control law (11) can be combined by means of Lemma 2; so the following

$$\Delta V = \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}^T}_{\mathcal{X}} \underbrace{\begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix}}_{\mathcal{Q}} \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}}_{\mathcal{X}} < 0 \quad \text{subject to} \quad \underbrace{\begin{bmatrix} A_h & -I_{n_x} & B_h \\ \left(G_{hhh^-}\right)^{-1} K_{hh^-} C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix}}_{\mathcal{W}} \underbrace{\begin{bmatrix} x_k \\ x_{k+1} \\ u_k \end{bmatrix}}_{\mathcal{X}} = 0 \qquad (19)$$

is equivalent to

$$\mathcal{M} \begin{bmatrix} A_h & -I_{n_x} & B_h \\ \left(G_{hhh^-}\right)^{-1} K_{hh^-} C_h & 0_{n_u \times n_x} & -I_{n_u} \end{bmatrix} + (*) + \begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0_{n_u} \end{bmatrix} < 0, \quad \mathcal{M} \in \mathbb{R}^{(2n_x+n_u)\times(n_x+n_u)}. \qquad (20)$$

Choosing $\mathcal{M} = \begin{bmatrix} 0_{n_x} & 0_{n_x \times n_u} \\ H_{hh^-} & \mathcal{J} G_{hhh^-} \\ 0_{n_u \times n_x} & G_{hhh^-} \end{bmatrix}$ and applying the relaxation Lemma 1 concludes the proof. ■

Note that Example 1 shows that, apparently, Theorem 2 is more relaxed that over both Lemma 3 [26] and Theorem 1; nonetheless this fact does not always hold. Let us now test all of them for the same selection of the matrix $\mathcal{J}$.

Example 2. Consider a TS model (3) with $r = 2$ and matrices

$$A_1 = \begin{bmatrix} 0.7 & 1.2 + 0.3a \\ -0.6 & -0.3 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & 1.6 \\ -1.4 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ -1.8 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -2.8 - 0.5b \end{bmatrix}, C_1 = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}^T, \text{and } C_2 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^T.$$

The real-valued parameters are defined as $a \in [-7, 9]$ and $b \in [-3.4, 2]$. Figure 2 shows the feasible sets for Lemma 3 (black-$\bullet$) and Theorem 2 (blue-□) with $\mathcal{J} = B_h$; both approaches consider a PDC control law (5). It can be seen that the feasibility set overlap.

As stated above, using a delayed membership functions in the Lyapunov function as well as non-PDC control law produces relaxed conditions. The feasibility region of Theorem 1 and Theorem 3 are plotted in Figure 3: effectively there is an improvement between Theorem 1 in comparison with Lemma 3 and

Theorem 3 in contrast with Theorem 2.

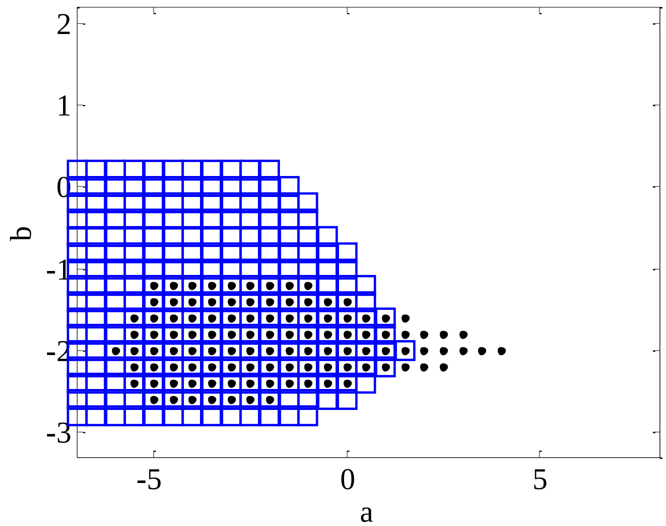

Figure 2. Solution set for conditions in Lemma 3 (black-•) and Theorem 2 (blue-□) with a PDC for Example 2.

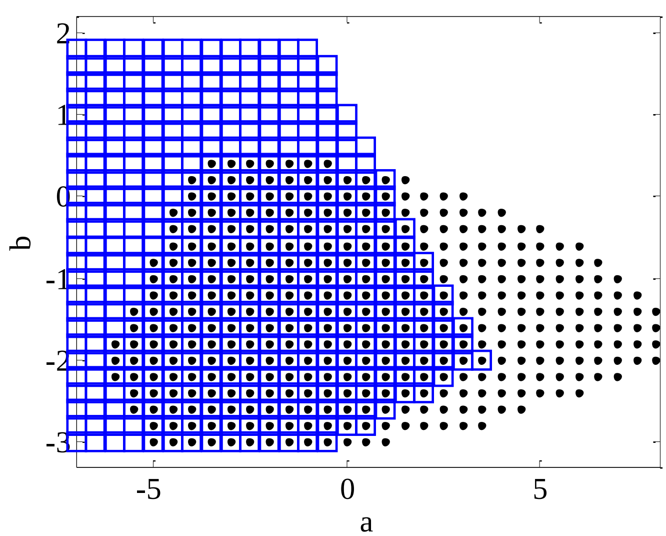

Figure 3. Solution set for conditions in Theorem 1 (black-•) and Theorem 3 (blue-□) with a delayed approach for Example 2. ♦

Note that the sets of feasible solutions *overlap*, i.e., none of the approaches is superior to the other. The question is whether or not both sets can be obtained via a unified LMI problem. The idea follows the work of [38]–[40] where, without adding complexity to the problem, a simple positive scalar ε chosen in a logarithmically spaced family of values $\varepsilon \in \{10^{-6}, 10^{-5}, \ldots, 10^{6}\}$ allows outperforming classical results obtained via Finsler's lemma. Therefore, the unification of the approaches uses this path and considers for (20) a matrix introducing ε: $\mathrm{M} = \begin{bmatrix} \varepsilon F_{hh^-} & \varepsilon \mathrm{J}\, G_{hhh^-} \\ H_{hh^-} & \mathrm{J}\, G_{hhh^-} \\ \varepsilon L_{hh^-} & G_{hhh^-} \end{bmatrix}$.

Thus it gives the following result:

Theorem 4. The nonlinear model (1) under the control law (11) has the origin asymptotically stable if there exist scalar $\varepsilon > 0$ and matrices $P_j = P_j^T > 0$, G_{ijl}, F_{jl}, H_{jl}, L_{jl}, and K_{jl} such that (4) holds with:

$$\Upsilon_{ij}^{l} = \begin{bmatrix} \Gamma^{(1,1)} & (*) & (*) \\ \Gamma^{(2,1)} & -H_{jl} - H_{jl}^{T} + P_j & (*) \\ \Gamma^{(3,1)} & \left(H_{jl} B_i - \mathrm{J}\, G_{ijl}\right)^{T} - \grave{o} L_{jl} & \grave{o} L_{jl} B_i - G_{ijl} + (*) \end{bmatrix}, \quad (i, j, l) \in \{1, 2, \ldots, r\}^{3}; \qquad (21)$$

with $\Gamma^{(1,1)} = \grave{o}\left(F_{jl} A_i + \mathrm{J}\, K_{jl} C_i\right) + (*) - P_l$, $\Gamma^{(2,1)} = H_{jl} A_i + \mathrm{J}\, K_{jl} C_i - \grave{o} F_{jl}^{T}$, and $\Gamma^{(3,1)} = \grave{o} L_{jl} A_i + \grave{o}\left(F_{jl} B_i - \mathrm{J}\, G_{ijl}\right)^{T} + K_{jl} C_i$.

Proof. It follows a similar path as Theorem 3. ■

Remark 3. The results in Theorem 4 generalize those in Lemma 3, Theorem 1, Theorem 2, and Theorem 3 under the same relaxation scheme. For example, Theorem 3 is obtained by taking $\varepsilon \to 0$ in Theorem 4. Moreover, Theorem 4 includes Theorem 1: consider (4) with (21), choose $\varepsilon = 1$, $F_{jl} = G_{1jl}$, and $L_{jl} = G_{2jl}$; the resulting expression is the same as the first three columns and rows of Theorem 1 (see Figure 4). Table 1 shows the numerical complexity of the approaches.

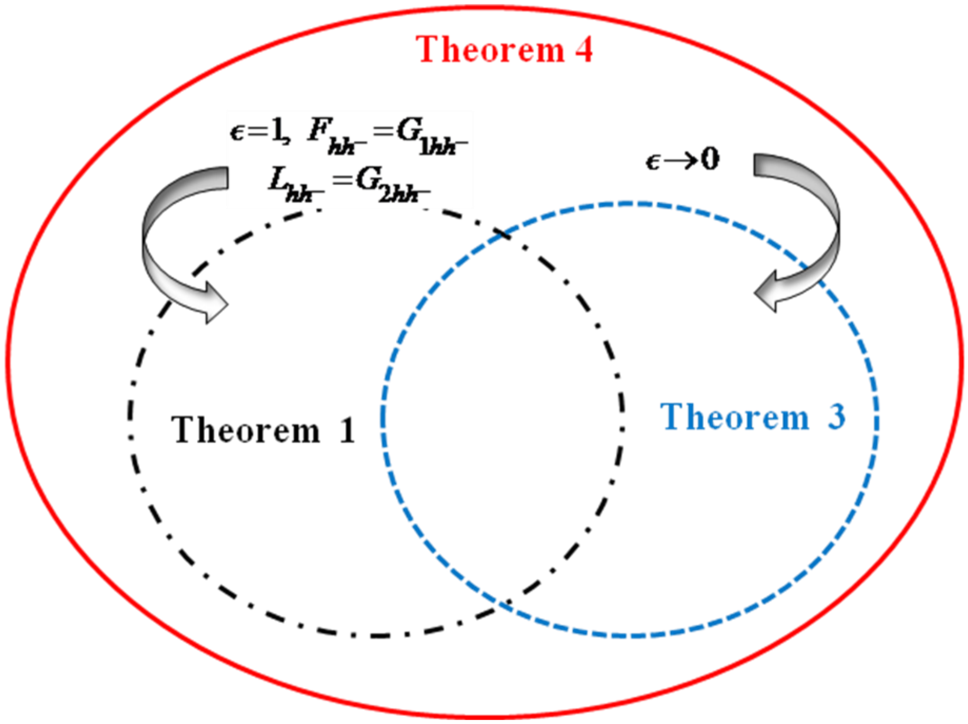

Figure 4. Graphic representation of Remark 3.

Table 1. Comparison of the numerical facts for the given approaches.

Approach	**Number of scalar decision variables** (N_d)	**Row size of the LMI problem** (N_l)
Lemma 3 [26] (PDC)	$r\begin{pmatrix}0.5n_x(n_x+1)+n_xn_y+n_xn_u\\ +2\left(rn_x^2+rn_xn_u\right)\\ +0.5n_u(n_u+1)\end{pmatrix}+n_u^2$	$2r^3(n_x+n_u)$ $+n_xr$
Theorem 1 (Delayed-non-PDC)	$r\begin{pmatrix}0.5n_x(n_x+1)+0.5n_u(n_u+1)\\ +2\left(rn_x^2+rn_xn_u\right)+rn_xn_y\\ +n_xn_u+r^2n_u^2\end{pmatrix}$	$2r^3(n_x+n_u)$ $+n_xr$
Theorem 2 (PDC)	$r\left(0.5n_x(n_x+1)+n_un_y\right)$ $+r^2n_x^2+n_u^2$	$r^3(2n_x+n_u)$ $+n_xr$
Theorem 3 (Delayed-non-PDC)	$r\begin{pmatrix}0.5n_x(n_x+1)+r^2n_u^2\\ +r\left(n_x^2+n_un_y\right)\end{pmatrix}$	$r^3(2n_x+n_u)$ $+n_xr$
Theorem 4 (Delayed-non-PDC)	$r\begin{pmatrix}0.5n_x(n_x+1)+r^2n_u^2\\ +r\left(2n_x^2+n_un_y+n_xn_u\right)\end{pmatrix}$	$r^3(2n_x+n_u)$ $+n_xr$

Example 2 (continued). Now, let us implement LMI conditions in Theorem 4 by selecting J $= B_h$. The feasible sets of solutions for Theorem 1 together with Theorem 3 are plotted in (black-•), and Theorem 4 (blue-□) are displayed in Figure 5. It illustrates how Theorem 4 overcomes Theorem 1 and Theorem 3. The numerical complexity of the approaches is proportional to the number of scalar decision variables (N_d) and the row size (N_l) of the LMI problem [41], [42], it can be approximated by $\log_{10}\left(N_d^3 N_l\right)$ [43]; thus for Theorem 1 is 7.3584, Theorem 3 is 6.2379 and for Theorem 4 is 6.9337.

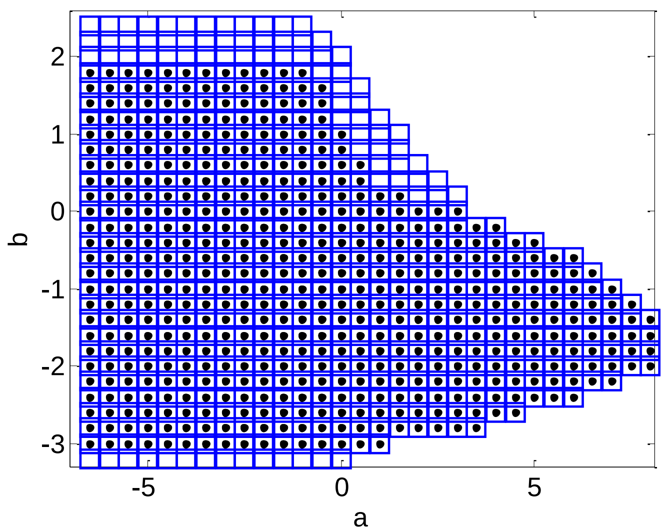

Figure 5. Solution set for conditions in Theorem 1 together with Theorem 3 (•) and Theorem 4 (□) with a delayed non-PDC control law for Example 2. ♦

The following example has been borrowed from [26], for comparison purposes a scalar $\beta > 0$ has been added.

Example 3. Consider the TS model (3) with the following vertex matrices [26]:

$$A_1 = \begin{bmatrix} 0.55 & 0.12 & 0.27 & 0.23 \\ 0.37 & 0.51 & -0.39 & 0.36 \\ -0.14 & -0.25 & 0.65 & 0.47+\beta \\ -0.53 & -0.15 & 0.22 & 0.46 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.62 & -0.29 & -0.31 & 0.28 \\ 0.24 & 0.59 & -0.23 & 0.19 \\ 0.19 & -0.37 & 0.43 & 0.15 \\ 0.16 & 0.31 & 0.22 & 0.55 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.4 \\ -0.4 \\ 1.5 \\ 1.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.25 \\ 0.20+\beta \\ -0.35 \\ 0.20 \end{bmatrix},$$

$C_1 = \begin{bmatrix} 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, and $C_2 = \begin{bmatrix} 0.41 & 0 & 0 & 0 \\ 0.5 & 0 & 0.7 & 0 \end{bmatrix}$; where $\beta > 0$ is a real-valued parameter. Testing the feasibility of the approaches hereby presented with $\mathrm{J} = B_h$, the following values have been obtained using SeDuMi [44] within YALMIP [45]:

a) Conditions in Lemma 3 [26] and Theorem 1 are feasible for $\beta = 0.103$;

b) Theorems 2 and 3 were found feasible for $\beta = 0.594$;

c) Conditions in Theorem 4 provide a solutions up to $\beta = 0.628$.

This example illustrates how LMI conditions in Theorem 4 outperform those in the existing literature.

IV. Conclusions and Discussions

An alternative SOFC design for nonlinear systems, *exactly* expressed as TS models, has been introduced. The main idea is based on how to choose a proper matrix M to satisfy the equivalence of the Finsler's lemma. The methodology takes full advantage of recent results on the field and overcomes previous ones in the literature. Its main interest is to “unify” several approaches in one simple result. Note that the conditions sum up to the *same* complexity level as it requires solving LMI problems of the same size but repeated several times according to a logarithmically spaced scalar. Several numerical examples are given in order to show the effectiveness of the proposed approaches.

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