Observer-based controller design for Takagi-Sugeno fuzzy systems with local nonlinearities

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Abstract— **This paper presents an observer-based controller design approach. To handle the inherent nonlinearities Takagi-Sugeno fuzzy modelling is used with nonlinear consequents. The purpose of using local nonlinear models is to reduce the number of fuzzy rules as well as to handle nonlinearities depending on unmeasured states. The design conditions are defined in form of Linear Matrix Inequalities, which can be efficiently solved. The obtained conditions are tested in simulation on an inverted pendulum model.**

I. Introduction

In many real-life applications there is no direct access to all the states, for instance because they cannot be explicitly measured or the costs of the sensors are too high. A state observer can be used to address this problem and estimate the unmeasured states. Among many options, the most commonly used are the Luenberger observer [10] and the Kalman filter [7]. Using the estimated states a state feedback controller can be designed to achieve the desired stability and performance of the system.

Usually the dynamic model of a system is nonlinear. Linear approximations are very common, however they provide only local conclusions [8]. In the last decades many approaches have been developed for handling nonlinearities, among which a very popular option is the Takagi Sugeno (TS) fuzzy modelling. TS models are convex combinations of local linear models blended by the so called membership functions, which depend on the scheduling or premise variables. TS models have the property that they can exactly represent a nonlinear model on a compact set.

TS fuzzy models can be used to design nonlinear observers [3]. In the case when the premise variables are measured the observer can easily be designed, but in many applications the premise variables depend on unmeasured states. This problem is usually solved by including a Lipschitz condition on the membership functions. However this condition is very conservative and preferably other design methods should be used [3]. An alternative method for nonlinear observer design is presented in [1], where all the nonlinearities are compressed into a nonlinear vector function, so that the rest of the dynamics are linear. In order to develop the results in [1] the nonlinear vector functions must fulfill a non-decreasing condition.

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We propose to use this approach in combination with TS fuzzy models, where some of the scheduling variables depend on unmeasured states. The main advantage of this method is that we can exploit the observer design methods available for the case when the scheduling variables are available and at the same time handle those that are not available. We provide an approach of observer-based controller design for nonlinear systems represented by TS fuzzy models. The nonlinearities are separated into two parts, where one is for nonlinearities with measured state variables and the other one is for nonlinearities with unmeasured states. The nonlinearities that depend on measured states are treated in the classical way, and the unmeasured-state-nonlinearities are handled using the approach in [4]. The separation of nonlinearities also reduces the number of rules necessary for the TS representation. Papers already exist in the literature on similar topics on separating the nonlinearities, see e.g. [5], [11], [12]. In [5] the problem of controller design is associated with TS fuzzy models with nonlinear consequent parts. In [11] a similar structure is used for a robust observer design. Observer-based controller design for systems with nonlinear consequent parts have been developed in [12]. In [12], it is assumed that the nonlinear functions are sector bounded, so the following condition is true

$$\phi_i(x(t)) \in \mathrm{co}\{0, E_i x(t)\}, \tag{1}$$

where $\phi_i(t)$ is the nonlinearity, $\mathrm{co}\{x, y\}$ is the convex hull of x, y. We address a different type of nonlinearity, which satisfies a non-decreasing condition. With this condition, for instance it is possible to include affine terms in the nonlinearities, which is not possible with (1). Another good example can be a nonlinear function which is not zero at $x = 0$, which cannot be defined with (1), but can fulfill the non-decreasing property.

Notations. Let $F = F^T \in R^{n\times n}$ be a real symmetric matrix, $F > 0$ and $F < 0$ mean that F is positive definite and negative definite, respectively. I denotes the identity matrix and 0 the zero matrix of appropriate dimensions. The symbol $*$ in a matrix indicates a transposed quantity in the symmetric position, for instance $\begin{pmatrix} P & * \\ A & P \end{pmatrix} = \begin{pmatrix} P & A^T \\ A & P \end{pmatrix}$, and $A + * = A + A^T$. The notation $\mathrm{diag}(f_1, ..., f_n)$, where $f_1, ..., f_n \in \mathbb{R}$, stands for the diagonal matrix, whose diagonal components are $f_1, ..., f_n$.

The rest of the paper is organized as follows, in Section II we introduce the necessary concepts for TS fuzzy models with nonlinear consequent parts. As well as we propose the structure for the observer and controller. Section III presents the main results together with the conditions developed for observer and controller design. The observer-based controller is illustrated on an example in Section IV and Section V concludes the paper.

II. Preliminaries and problem statement

The classic TS fuzzy model is a convex combination of linear models, having the form:

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{s} h_i(z)(A_i x + B_i u) \\ y &= \sum_{i=1}^{s} h_i(z) C_i x,\end{aligned} \tag{2}$$

where x is the state vector, u is the control input, s is the number of rules, z is the premise vector, and h_i, $i = 1, ..., s$ are nonlinear functions with the property

$$h_i \in [0,\, 1],\, i = 1, ..., s, \quad \sum_{i=1}^{s} h_i(z) = 1. \tag{3}$$

These nonlinear functions are called the membership functions. Matrices A_i, B_i, and C_i represent the $i-th$ local model. Throughout this paper, the following shorthand notation is used to represent convex sums of matrix expressions:

$$F_z = \sum_{i=1}^{s} h_i(z) F_i. \tag{4}$$

Based on this notation, (2) can be rewritten as

$$\begin{aligned}\dot{x} =& A_z x + B_z u \\ y =& C_z x.\end{aligned} \tag{5}$$

In order to develop our results we will use the following lemmas.

Lemma 1 (Congruence): Given matrix $P = P^T$ and a full column rank matrix Q it holds that

$$P > 0 \quad \Rightarrow \quad QPQ^T > 0.$$

Estimation and control problems are often defined as a double sum negativity problem having the form

$$\sum_{i=1}^{s} h_i(z) h_j(z) F_{ij} < 0, \tag{6}$$

with symmetric matrices F_{ij}, and nonlinear functions h_i, where $i = 1, ..., s$, satisfying the convex sum property (3).

Lemma 2 ([13]): Equation (6) is satisfied if the following conditions hold

$$\begin{aligned} F_{ii} <& 0 \\ \frac{2}{s-1} F_{ii} + F_{ij} + F_{ji} <& 0 \quad \forall i, j = 1, ..., s\end{aligned} \tag{7}$$

In the following we consider the model structure

$$\begin{aligned}\dot{x} =& A_z x + B_z u + B_z G_z \psi(Hx) \\ y =& C_z x,\end{aligned} \tag{8}$$

where $x \in \mathbb{R}^{n_x}$ represents the state, $u \in \mathbb{R}^{n_u}$ stands for the control input, $y \in \mathbb{R}^{n_y}$ is the measured output; A_z, B_z, G_z, and C_z are convex combination of matrices as in (4). We assume that the scheduling vector z only depends on measured variables. In fact, the form (8) allows us to avoid - for a class of nonlinear systems - having scheduling variables dependent on unmeasured states. The nonlinearities that contain unmeasured states are collected in the vector function $\psi()$.

A somewhat restrictive assumption we make is on the form of the unmeasured nonlinear part, i.e. $B_z G_z \psi$. Note however that such a form often appears e.g. for mecahnical systems in classical state-space form obtained from Euler-Lagrange equations. To see this, let us consider the model of a robot arm

$$M(\theta)\ddot{\theta} = -F(\theta, \dot{\theta}) - G(\theta) + \tau, \tag{9}$$

where τ represents the torque; θ, $\dot{\theta}$ and $\ddot{\theta}$ are the angles, angular velocities and angular accelerations. $M(\theta)$ is the mass matrix, $F(\theta, \dot{\theta})$ contains the Centrifugal and Coriolis matrices and $G(\theta)$ is the gravity matrix. In order to obtain a classical state-space representation, the whole equation must be multiplied with the inverse of the mass matrix. In this context B_z is $M(\theta)^{-1}$.

The $\psi(Hx) \in \mathbb{R}^r$ is an r-dimensional vector where $H \in \mathbb{R}^{r \times n_x}$ and each entry is a function of a linear combination of the states, i.e.

$$\psi_i = \psi_i(\sum_{j=1}^{n} H_{ij} x_j), \quad i = 1, ..., r.$$

To develop our results, the vector ψ must fulfill the following assumption.

Assumption 1: For any $i \in \{1, ..., r\}$ there exist constants $0 < b_i \leq \infty$, so that

$$0 \leq \frac{\psi_i(v) - \psi_i(w)}{v - w} \leq b_i, \quad \forall v, w \in \mathbb{R}, v \neq w. \tag{10}$$

As a remark, let us consider the case when the nonlinearities do not satisfy (10), but the following is true:

Assumption 2: For any $i \in \{1, ..., r\}$ there exist constants $0 \leq a_i < b_i \leq \infty$, so that

$$a_i \leq \frac{\psi_i(v) - \psi_i(w)}{v - w} \leq b_i, \quad \forall v, w \in \mathbb{R}, v \neq w. \tag{11}$$

If $a_i \neq 0$ a new functions can be defined $\tilde{\psi}_i(v) := \psi_i(v) - a_i v$, which satisfy (10), with $\tilde{a}_i = 0$, and $\tilde{b}_i = b - a$, and the new terms are added in the A_z matrix. Assumption 2 intuitively bounds the rate of change of the nonlinearity, and corresponds to a global Lipschitz property of ψ, when b_i is finite and ψ_i is continuosly differentiable. This assumption is made in [1], [2], [4], [6].

As in [2], in view of (10), there exist $\delta_i(t) \in [0,\, b_i]$, so that for any $v,\, w \in \mathbb{R}$

$$\psi_i(v) - \psi_i(w) = \delta_i(t)(v - w). \tag{12}$$

Let $\delta(t) = \mathrm{diag}(\delta_1(t), ..., \delta_r(t))$. Note that this condition althought similar, is not equivalent to (1). Different type of nonlinearities can be treated with (10) than with (1). For example let us consider the following nonlinear functions,

$$\begin{aligned}\psi_1(x) =& x^2 - 2x + 3, \quad x \in [-1, 1] \\ \psi_2(x) =& e^x, \quad x \in [0, 2]\end{aligned} \tag{13}$$

These nonlinear functions fulfill (10), but the convex hull condition, defined in (1), cannot be fulfilled.

In order to develop our results the following observer, similar to the one in [12], is considered

$$\begin{aligned}\dot{\hat{x}} =& A_z\hat{x} + B_z u + B_z G_z \psi(H\hat{x} + L_\psi(y - C\hat{x})) + L_z(y - \hat{y}) \\ y =& C_z\hat{x},\end{aligned} \tag{14}$$

where $\hat{x}$ denotes the estimate of x. The observer gains are stored in L_z; L_ψ is an injection term to obtain a less conservative design.

The control law to stabilize this system has the following form,

$$u = -K_z\hat{x} - G_z\psi(H\hat{x} + L_\psi(y - C\hat{x})), \tag{15}$$

where K_z is the fuzzy controller gain. For the controller the estimated states are used. The main objective is to obtain an observer-based controller, i.e. we want to stabilize the system at the origin using an observer-based state feedback control. Let us consider the error dynamics, $e = x - \hat{x}$, from where we obtain

$$\begin{aligned}\dot{e} =& (A_z - L_z C_z)e \\ &+ B_z G_z \left(\psi(Hx) - \psi(H\hat{x} + L_\psi(y - C\hat{x}))\right)\end{aligned} \tag{16}$$

Based on Assumption 1 we can rewrite (16) in the form

$$\begin{aligned}\dot{e} =& (A_z - L_z C_z)e + B_z G_z \delta(t)\eta \\ \eta =& (H + L_\psi C)e.\end{aligned} \tag{17}$$

Finally, the closed loop dynamics are

$$\begin{aligned}\dot{e} =& (A_z - L_z C_z)e + B_z G_z \delta(t)\eta \\ \dot{x} =& (A_z - B_z K_z)x + B_z K_z e + B_z G_z \delta(t)\eta \\ \eta =& (H + L_\psi C)e.\end{aligned} \tag{18}$$

III. Main results

To develop the stabilization conditions, instead of considering $\dot{e}$ and $\dot{x}$ as in (18) we employ $\dot{\hat{x}}$. In this way we obtain an easier design approach. For $\hat{x}$ we have the following dynamics after applying the control law from (15),

$$\begin{aligned}\dot{\hat{x}} =& A_z\hat{x} + B_z u + B_z G_z \psi(H\hat{x} + L_\psi C_z e) + L_z C_z e \\ =& A_z\hat{x} + B_z(-K_z\hat{x} - G_z\psi(H\hat{x} + L_\psi C_z e)) \\ &+ B_z G_z \psi(H\hat{x} + L_\psi C_z e) + L_z C_z e \\ =& (A_z - B_z K_z)\hat{x} - B_z G_z \psi(H\hat{x} + L_\psi(y - C\hat{x})) + \\ & B_z G_z \psi(H\hat{x} + L_\psi(y - C\hat{x})) + L_z C_z e \\ =& (A_z - B_z K_z)\hat{x} + L_z C_z e.\end{aligned} \tag{19}$$

If e and $\hat{x}$ are converging to 0, then also x is converging to 0. We denote the augmented system states with $\hat{x}_a := \begin{bmatrix}\hat{x} & e\end{bmatrix}^T$, and the dynamics has the form,

$$\begin{aligned}\begin{bmatrix}\dot{\hat{x}} \\ \dot{e}\end{bmatrix} =& \begin{bmatrix}A_z - B_z K_z & L_z C_z \\ 0 & A_z - L_z C_z\end{bmatrix}\begin{bmatrix}\hat{x} \\ e\end{bmatrix} + \begin{bmatrix}0 \\ B_z G_z\end{bmatrix}\delta(t)\eta \\ \eta =& (H + L_\psi C)e.\end{aligned} \tag{20}$$

The augmented system can be considered as a cascaded system which was well studied in [9]. It can be seen that an observer can be design for (17), so that the error dynamics will be globally asymptotically stable at the origin, without the use of the controller. Based on [9], if the dynamics of (19) without the $L_z C_z e$ term converging to 0 then also the augmented system in (20) will be stable at the origin. The following theorems summarize this idea.

Theorem 1: Consider system (17) and assume that the system states in (8) do not have a finite escape time. If there exist $P_1 = P_1^T > 0$, $M = M^T = \mathrm{diag}(m_1, .., m_r) > 0$, N_i, W_ψ and constant ϵ so that

$$\begin{aligned}\Theta_{ii} <& 0 \\ \frac{2}{s-1}\Theta_{ii} + \Theta_{ij} + \Theta_{ji} <& 0\end{aligned} \tag{21}$$

then (17) is globally asymptotically stable at the origin. The notation Θ_{ij} is defined as

$$\Theta_{ij} = \begin{bmatrix}\mathcal{A}(P_1, N_i, C_j, \epsilon) & \mathcal{B}(P_1, M, W_\psi) \\ * & \nu(M)\end{bmatrix}. \tag{22}$$

with $\mathcal{A}(P_1, N_i, \epsilon) = P_1 A_i - N_i C_j + * + \epsilon I$, $\mathcal{B}(P_1, M, T_\psi) = P_1 B_i G_i + H^T M + C_i^T W_\psi^T$ and $\nu(M) = -2M\mathrm{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r})$, where b_i, for all $i = 1, ..., r$ are defined in (10).

Proof: Let us consider the Lyapunov function candidate $V(e) = e^T P_1 e$. The proof for the error dynamics follows the lines of the proof of Theorem 2 from [4], which we extend for TS fuzzy systems. The derivative of the Lyapunov function candidate along the trajectories of e will be

$$\begin{aligned}&\langle \nabla V(e), (A_z - L_z C_z)e + B_z G_z \delta(t)\eta) \rangle = \\ &e^T(P_1(A_z - L_z C_z) + *)e + 2e^T P_1 B_z G_z \delta(t)\eta\end{aligned} \tag{23}$$

In view of Lemma 2 and (22) we obtain

$$\begin{aligned}&\langle \nabla V(e), (A_z - L_z C_z)e + B_z G_z \delta(t)\eta) \rangle \leq \\ &- e^T \epsilon I e - 2e^T(H^T M + C_z^T V_\psi^T)\delta(t)\eta - \eta^T \delta(t)^T \nu(M)\delta(t)\eta\end{aligned} \tag{24}$$

Let us denote $W_\psi := ML_\psi$ to avoid Bilinear Matrix Inequalities. By using $\eta = (H + L_\psi C_z)e$ from (17) the following is obtained:

$$\begin{aligned}&\langle \nabla V(e), (A_z - L_z C_z)e + B_z G_z \delta(t)\eta) \rangle \leq \\ &- \epsilon\|e\|^2 - 2\eta^T \left(M\delta(t) - \delta(t)^T M \mathrm{diag}(\tfrac{1}{b_1}, ..., \tfrac{1}{b_r})\delta(t)\right)\eta.\end{aligned} \tag{25}$$

Let us consider only this part of the equation:

$$f_M = M\delta(t) - \delta(t)^T M \mathrm{diag}(\frac{1}{b_1}, ..., \frac{1}{b_r})\delta(t). \tag{26}$$

Since all the elements are in a diagonal form we can examine them element by element, which leads to: $\delta_i(t)m_i\left(1 - \delta_i(t)\frac{1}{b_i}\right)$. From (10) we know that $\delta_i \in [0, b_i]$ and $m_i > 0$, so we can conclude that $f_M \geq 0$. Finally we obtain,

$$\langle \nabla V(e), (A_z - L_z C_z)e + B_z G_z \delta(t)\eta) \rangle \leq -\epsilon\|e\|^2, \tag{27}$$

so the error dynamics is globally asymptotically stable at the origin. ■

For the controller design the following result is obtained.

Theorem 2: Consider system (20), and assume that there already exists an observer with parameters L_i and L_ψ. If there exist $X = X^T > 0$, R_i, so that

$$\begin{aligned} \Phi_{ii} &< 0 \\ \frac{2}{s-1}\Phi_{ii} + \Phi_{ij} + \Phi_{ji} &< 0, \end{aligned} \tag{28}$$

then (20) is globally asymptotically stable at the origin, where Φ_{ij} is defined as

$$\Phi_{ij} = A_i X - B_i R_j + *. \tag{29}$$

Proof: Let us consider the Lyapunov function candidate $V_x(\hat{x}) = \hat{x}^T X^{-1} \hat{x}$. We differentiate along the trajectories of $\hat{x}$

$$\begin{aligned} &\langle \nabla V_x(\hat{x}), (A_z - B_z K_z)\hat{x}) \rangle = \\ &\hat{x}^T (X^{-1} A_z + X^{-1} B_z K_z + *)\hat{x} \end{aligned} \tag{30}$$

From here by using Lemma 1 the following is obtained

$$A_z X + B_z K_z X + * < 0. \tag{31}$$

By denoting $R_z = K_z X$ we obtain Φ_{ij}. It was assumed that there exists an observer which fulfills Theorem 1, so at this point it was proved that the two individual systems are stable on their own, without any interconnection between them. Based on [9], since (17) is globally asymptotically stable (GAS) and (19) without the $L_z C_z e$ term is globally asymptotically stable, then the augmented system in (20) is also GAS. ∎

In many real-life applications disturbances can appear on the system and their effects need to be attenuated. In order to do this an H_∞ approach can be used. Let us consider the following system

$$\dot{x} = A_z x + B_z u + B_z G_z \psi(Hx) + S_z d, \tag{32}$$

where $d \in \mathbb{R}^{n_d}$ is the disturbance, and S_z is a fuzzy matrix. In the next Corollary we give conditions to find the H_∞ index for a given system with the controller and observer calculated using Theorem 1 and 2. The problem of maximizing the H_∞ index will be addressed in future works.

Corollary 1: Consider system (32), with observer (14), and thus the closed loop system with controller (15) is globally asymptotically stable in the absence of the perturbation. With the observer and controller gains computed using Theorem 1 and 2, the attenuation of the disturbance is at least γ, if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $M = \text{diag}(m_1, ..., m_r)$ and ϵ so that

$$\begin{aligned} \Delta_{ii} &< 0 \\ \frac{2}{s-1}\Delta_{ii} + \Delta_{ij} + \Delta_{ji} &< 0. \end{aligned} \tag{33}$$

The notation

$$\Delta_{ij} = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} & 0 & 0 \\ * & \mathcal{D}_{22} & \mathcal{D}_{23} & P_1 S_i \\ * & * & \nu(M) & 0 \\ * & * & * & -\gamma^2 I \end{bmatrix} \tag{34}$$

where

$$\begin{aligned} \mathcal{D}_{11} &= P2(A_i - B_i K_j) + * + D^T D \\ \mathcal{D}_{12} &= P_2 L_i C_j + D^T D \\ \mathcal{D}_{22} &= P_1(A_i - L_i C_j) + * + \epsilon I + D^T D \\ \mathcal{D}_{23} &= P_1 B_i G_j + H^T M + C_i^T L_\psi^T M^T. \end{aligned} \tag{35}$$

IV. Numerical example

We illustrate the application of Theorem 1 and 2 on an example. Let us consider the following nonlinear model of an inverted pendulum on a cart adapted from [12]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-dx_2 - a(mlx_2)^2 \sin(x_1)\cos(x_1) + mgl\sin(x_1)}{\alpha(x_1)} \\ &\quad + \frac{-aml\cos(x_1)}{\alpha(x_1)}\tilde{u} \\ y &= x_1 \end{aligned} \tag{36}$$

where $\alpha(x_1) = (J + ml^2) - a(ml\cos(x_1))^2$. This can be transformed into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{\rho}_1(x_1)x_2 + \rho_2(x_1)\rho_3(x_1)\tilde{\psi}(Hx) \\ &\quad + \rho_2(x_1)\beta(x_1) + \rho_2(x_1)\tilde{u} \end{aligned} \tag{37}$$

where

$$\begin{aligned} \tilde{\rho}_1(x_1) &= -\frac{d}{\alpha(x_1)}, \quad \rho_2(x_1) = \frac{-aml\cos(x_1)}{\alpha(x_1)} \\ \rho_3(x_1) &= ml\sin(x_1), \quad \beta(x_1) = -\frac{g\sin(x_1)}{a\cos(x_1)} \\ \tilde{\psi}(Hx) &= x_2^2, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned} \tag{38}$$

Since $\beta(x_1)$ depends only on x_1 which is measured, we can eliminate this term by replacing

$$\tilde{u} = u - \beta(x_1) \tag{39}$$

So the new system will be

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{\rho}_1(x_1)x_2 + \rho_2(x_1)\rho_3(x_1)\tilde{\psi}(Hx) + \rho_2(x_1)u \\ y &= x_1 \end{aligned} \tag{40}$$

Let $x_2 \in [-\sigma, \sigma]$, which leads to:

$$-2\sigma \leq \frac{\tilde{\psi}(v) - \tilde{\psi}(w)}{v - w} \leq 2\sigma. \tag{41}$$

Since the nonlinearity does not fulfill Assumption 1, we apply the following modifications:

$$\begin{aligned} \psi(Hx) &= \tilde{\psi}(Hx) + 2\sigma x_2 = x_2^2 + 2\sigma x_2 \\ \rho_1(x_1) &= \tilde{\rho}_1(x_1) - 2\sigma\rho_2(x_1)\rho_3(x_1) \\ &= \frac{-d + 2\sigma aml\cos(x_1)\sin(x_1)}{\alpha(x_1)} \end{aligned} \tag{42}$$

With these modifications the nonlinearity, $\psi(Hx)$, fulfills Assumption 1 with $b = 4\sigma$. The final form of the equation is:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \rho_1(x_1)x_2 + \rho_2(x_1)\rho_3(x_1)\psi(Hx) + \rho_2(x_1)u. \end{aligned} \tag{43}$$

Equation (43) has a form similar to (8). The model parameters were obtained from [12], and can be found in Table I. In order to obtain the exact fuzzy model for (43), we

TABLE I
PARAMETER TABLE

Notation	Value	Description
g $[m^s/s]$	9.8	gravitational acceleration
m [kg]	0.3	mass of pendulum
M [kg]	15	mass of cart
d [N/rad/s]	0.0007	friction coefficient
l [m]	0.3	length of pendulum
J $[kg\, m^2]$	0.3	moment of inertia
σ [rad/s]	4	max angular velocity

use the sector nonlinearity approach which leads to 8 rules. To determine these fuzzy rules we define a bound on the angle, $x_1 \in [-\frac{\pi}{3}, \frac{\pi}{3}]$; the same bound was used also in [12]. Due to its complicated form the membership functions, h_i, $i = 1, ..., r$, are not presented here, but some of the local models are:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ 0 & -0.96 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ -0.18 \end{bmatrix}, G_1 = -0.07 \\ A_8 &= \begin{bmatrix} 0 & 1 \\ 0 & 0.52 \end{bmatrix}, B_8 = \begin{bmatrix} 0 \\ -0.09 \end{bmatrix}, G_8 = 0.07 \end{aligned} \tag{44}$$

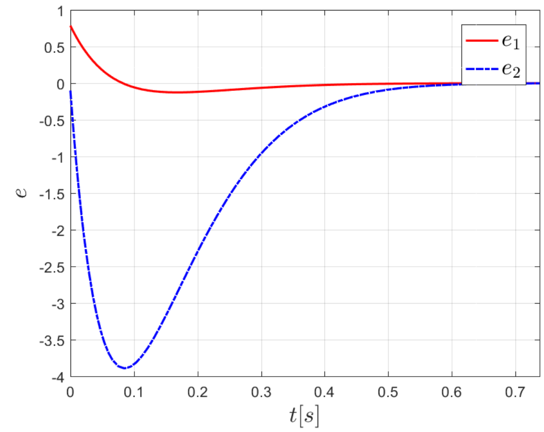
By applying Theorem 1 for this model, we obtain the following observer gains:

$$\begin{aligned} L_\psi &= 4.32\, 10^{-5}, L_1 = \begin{bmatrix} 27.48 \\ 182.13 \end{bmatrix}, L_2 = \begin{bmatrix} 28.13 \\ 186.41 \end{bmatrix} \\ L_3 &= \begin{bmatrix} 27.52 \\ 182.38 \end{bmatrix}, L_4 = \begin{bmatrix} 28.1 \\ 186.22 \end{bmatrix}, L_5 = \begin{bmatrix} 22.15 \\ 146.65 \end{bmatrix} \\ L_6 &= \begin{bmatrix} 22.79 \\ 150.9 \end{bmatrix}, L_7 = \begin{bmatrix} 22.19 \\ 146.92 \end{bmatrix}, L_8 = \begin{bmatrix} 22.76 \\ 150.72 \end{bmatrix}, \end{aligned} \tag{45}$$

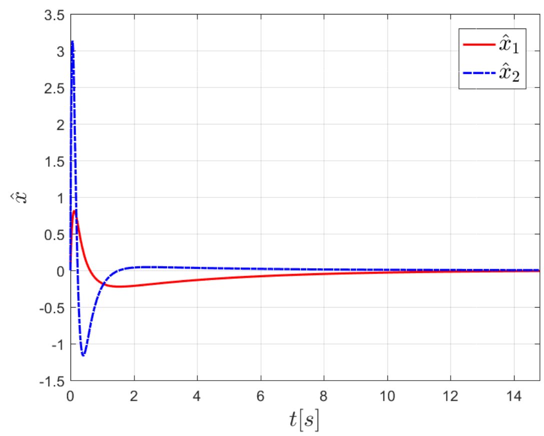
and controller gains,

$$\begin{aligned} K_1 &= \begin{bmatrix} -3.81 & -11 \end{bmatrix}, K_2 = \begin{bmatrix} -3.81 & -11 \end{bmatrix} \\ K_3 &= \begin{bmatrix} -6.52 & -21.48 \end{bmatrix}, K_4 = \begin{bmatrix} -6.52 & -21.47 \end{bmatrix} \\ K_5 &= \begin{bmatrix} -3.74 & -20.16 \end{bmatrix}, K_6 = \begin{bmatrix} -3.74 & -20.16 \end{bmatrix} \\ K_7 &= \begin{bmatrix} -6.73 & -32.71 \end{bmatrix}, K_8 = \begin{bmatrix} -6.73 & -32.71 \end{bmatrix}. \end{aligned} \tag{46}$$

We simulate our model for initial condition $x_0 = [\frac{\pi}{4}, -0.1]^T$, and the observer starts at initial condition $\hat{x}_0 = [0, 0]^T$. The evolution of the error dynamics and the observer states can be seen on Fig. 1(a) and Fig. 1(b). As it can be seen on Fig 1(a) the error dynamics stabilizes relatively fast, compared to the estimated states, but after $10s$ also the $\hat{x}$ is relatively close to the desired position. We also considered the perturbed dynamics described in (32) with $S = \text{diag}(0.05, 0.05)$ and the important state is the output so $D = [1, 0]$. The H_∞ index calculated for the obtained observer and controller using 1, and the smallest value obtained is $\gamma = 0.7$, so the attenuation of the pertubation is at least γ. To exemplify this effect we used the same perturbation on both states. The perturbation was a repeating stair sequence, which can be seen on Fig 2(a) and the resulting state dynamics can be seen on Fig. 2(b).



(a) Evolution of the estimation error



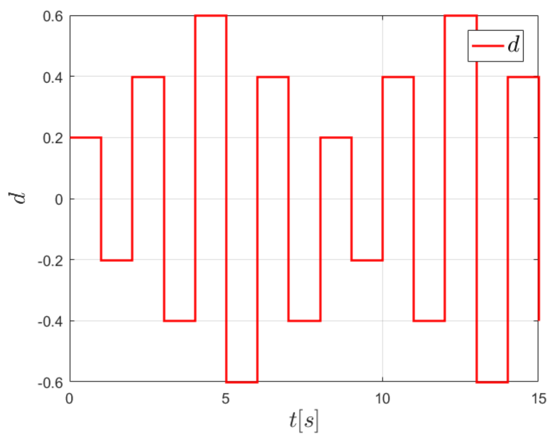
(b) Evolution of the estimated state vector

Fig. 1. Simulation results without disturbances

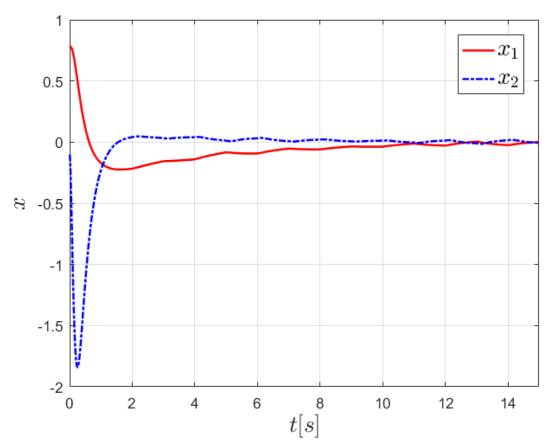
V. CONCLUSIONS AND FUTURE WORK

In this paper a nonlinear observer-based controller design approach was presented using TS fuzzy models with nonlinear consequent parts. First we showed the main differences of our approach compared to the existing results, highlighting the novelty of this paper. We then presented an observer design method to estimate the unknown states of the nonlinear system, and afterwards the controller was designed based on the estimated states. In our future work we will consider disturbances acting on the system. We have already given a method to calculate the H_∞ index. To illustrate the usage of the conditions, an inverted pendulum model was considered, and the simulations provided good results, also in the presence of the perturbations.

In future work we want to extend the type of nonlinearities which can be treated with this method, allowing for instance nonlinearities that are coming from multiplying two states. Another possible future work is to calculate the observer and controller gains so that the H_∞ attenuation is maximized. Finally, we would like to implement this approach on a real application.



(a) Disturbances applied to inverted pendulum



(b) Evolution of the state vector

Fig. 2. Simulation results with disturbances

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