# Analysis and design for periodic systems

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# Motivation

## Periodic systems

- nonlinear models with periodic parameters
- automotive, aerospace, industrial processes
- continuous and linear

## TS fuzzy models

- convex combination of linear models
- well-established methods
- conditions based on LMI

**This talk**

Analysis and design – discrete-time periodic TS models

# Outline

# Periodic TS model

Discrete-time periodic TS model:

$$\boldsymbol{x}(k+1) = \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k))(A_{j,i}\boldsymbol{x}(k) + B_{j,i}\boldsymbol{u}(k))$$

$$\boldsymbol{y}(k) = \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k))C_{j,i}\boldsymbol{x}(k)$$

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = C_{j,z}\boldsymbol{x}(k)$$

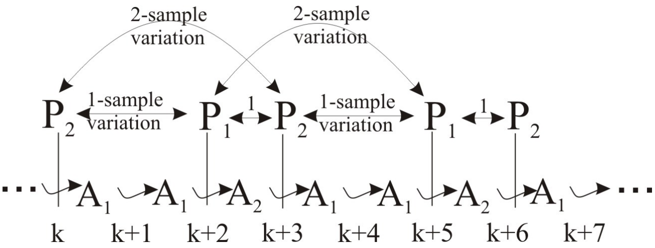
Periodicity: switching in a predefined sequence

Periods: $p_1,\ p_2,\ \ldots,\ p_{n_s},\ p_1$

# Periodic Lyapunov function

Periodic Lyapunov function: defined in the switching instants

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P_{j,z} \boldsymbol{x}(k)$$



# An example

TS subsystem

$$A_{11} = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} -0.5 & 0.2 \\ 0.4 & 0.9 \end{pmatrix}$$

Linear subsystem

$$A_2 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & a \end{pmatrix} \qquad a \in [-2, 2]$$

## Stability

- Quadratic Lyapunov function – none
- Non-quadratic, switching Lyapunov function – none
- Periodic Lyapunov function – $a \in [-1.2, 1.1]$

# Analysis

- Problem:

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k)$$

- $n_s$ subsystems
- $p_1, p_2, \ldots, p_{ns}$ periods

**Periodic Lyapunov function:**

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P_{j,z}\boldsymbol{x}(k)$$

# Conditions

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ M_{\underline{j+1},z}A_{\underline{j+1},z} & -M_{\underline{j+1},z} + (*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_{\underline{j+1},z+p_{\underline{j+1}}-1}A_{\underline{j+1},z+p_{\underline{j+1}}-1} & \Omega_{j+1,j+1} \end{pmatrix} < 0$$

$$\Omega_{j+1,j+1} = -M_{\underline{j+1},z+p_{\underline{j+1}}-1} + (*) + P_{\underline{j+1},z+p_{\underline{j+1}}}$$

Sufficient LMI conditions

# Example

- 2 subsystems
- periods: $p_1 = 2$, $p_2 = 2$

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k)) A_{1i} \boldsymbol{x}(k) \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k)) A_{2i} \boldsymbol{x}(k) \end{cases}$$

# Example – cont'd

$$A_{11} = \begin{pmatrix} -0.44 & -0.26 \\ -0.65 & 0.62 \end{pmatrix} \quad A_{12} = \begin{pmatrix} 1.1 & -0.2 \\ 0.53 & -0.27 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0.02 & 0.6 \\ -0.22 & -0.44 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 0.32 & -0.15 \\ -1 & 0.8 \end{pmatrix}$$

$h_{11} = exp(-x_1^2)$, $h_{12} = 1 - h_{11}$
$h_{21} = cos(x_1)^2$, $h_{22} = 1 - h_{21}$

$A_{12}$ and $A_{22}$ are unstable

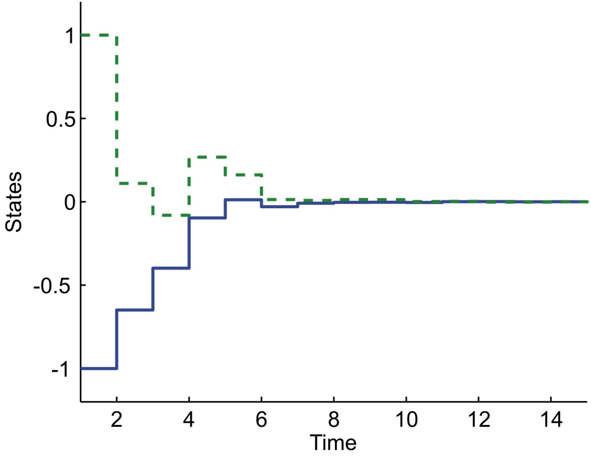
# Conditions

$$\begin{pmatrix} -P_{1,z} & (*) & (*) \\ M_{2,z}A_{2,z} & -M_{2,z}+(*) & (*) \\ 0 & M_{2,z+1}A_{2,z+1} & -M_{2,z+1}+(*)+P_{2,z+2} \end{pmatrix} < 0$$

$$\begin{pmatrix} -P_{2,z} & (*) & (*) \\ M_{1,z}A_{1,z} & -M_{1,z}+(*) & (*) \\ 0 & M_{1,z+1}A_{1,z+1} & -M_{1,z+1}+(*)+P_{1,z+2} \end{pmatrix} < 0$$

Feasible LMIs

# A trajectory



# Design problem

Periodic TS model:

$$\begin{aligned}\boldsymbol{x}(k+1) &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k)\\ \boldsymbol{y}(k) &= C_{j,z}\boldsymbol{x}(k)\end{aligned}$$

- Scheduling variables do not depend on the estimated states

Observer

$$\begin{aligned}\boldsymbol{x}(k+1) &= A_{j,z}\widehat{\boldsymbol{x}}(k) + B_{j,z}\boldsymbol{u}(k) + M_{j,z}^{-1}L_{j,z}(\boldsymbol{y} - \widehat{\boldsymbol{y}})\\ \widehat{\boldsymbol{y}}(k) &= C_{j,z}\widehat{\boldsymbol{x}}(k)\end{aligned}$$

# Estimation error

Estimation error:

$$\boldsymbol{e}(k+1) = (A_{j,z} - M_{j,z}^{-1} L_{j,z} C_{j,z}) \boldsymbol{e}(k)$$

Periodic Lyapunov function:

$$V(\boldsymbol{e}(k)) = \boldsymbol{e}(k)^T P_{j,z} \boldsymbol{e}(k)$$

# Conditions

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{0,a} & \Omega_{0,b} & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{p_{\underline{j+1}},a} & \begin{pmatrix} \Omega_{p_{\underline{j+1}},b} \\ +P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \end{pmatrix} < 0$$

where

$$\Omega_{l,a} = M_{\underline{j+1},z+l} A_{\underline{j+1},z+l} - L_{\underline{j+1},z+l} C_{\underline{j+1},z+l}$$
$$\Omega_{l,b} = -M_{\underline{j+1},z+l} + (*)$$

**Sufficient LMI conditions exist**

**Extension: $\alpha$-sample variation**

# Conditions – on an example

- 2 subsystems
- periods: $p_1 = 2$, $p_2 = 1$

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k)) A_{1i} \boldsymbol{x}(k) \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k)) A_{2i} \boldsymbol{x}(k) \end{cases}$$

$$\boldsymbol{y}(k) = \begin{cases} \sum_{i=1}^{r_1} h_{1i}(\boldsymbol{z}_1(k)) C_1 \boldsymbol{x}(k) \\ \sum_{i=1}^{r_2} h_{2i}(\boldsymbol{z}_2(k)) C_2 \boldsymbol{x}(k) \end{cases}$$

# Conditions

$$\begin{pmatrix} -P_{2,z} & (*) & (*) \\ M_{1,z}A_{1,z} - L_{1,z}C_{1,z} & -M_{1,z} + (*) & (*) \\ 0 & M_{1,z+1}A_{1,z+1} - L_{1,z+1}C_{1,z+1} & -M_{1,z+1} + (*) + P_{1,z+2} \end{pmatrix} < 0$$

$$\begin{pmatrix} -P_{1,z} & (*) \\ M_{2,z}A_{2,z} - L_{2,z}C_{2,z} & -M_{2,z} + (*) + P_{2,z+1} \end{pmatrix} < 0$$

# Example

$$A_{11} = \begin{pmatrix} 0.80 & 0.22 \\ -0.09 & 0.32 \end{pmatrix} \quad A_{12} = \begin{pmatrix} -0.82 & -0.44 \\ -1.25 & 0.33 \end{pmatrix}$$

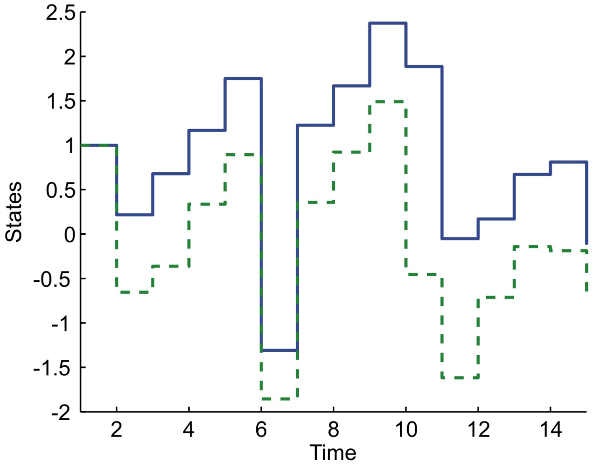
$$A_{21} = \begin{pmatrix} 0.44 & 0.46 \\ 0.93 & 0.41 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 0.84 & 0.20 \\ 0.52 & 0.67 \end{pmatrix}$$

$$C_{11} = C_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad C_{21} = C_{22} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

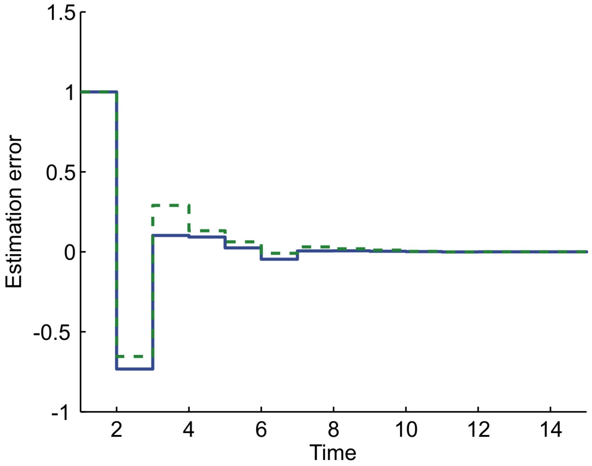
$$B = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$$

First subsystem not observable

# Trajectory of the states



# Estimation error



# Design problem

Periodic TS model:

$$\boldsymbol{x}(k+1) = A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k)$$

Controller:

$$\boldsymbol{u}(k) = -F_{j,z}H_{j,z}^{-1}\boldsymbol{x}(k)$$

Closed-loop system:

$$\boldsymbol{x}(k+1) = (A_{j,z} - B_{j,z}F_{j,z}H_{j,z}^{-1})\boldsymbol{x}(k)$$

# Conditions

$$\begin{pmatrix} -H_{\underline{j+1},z} - H^T_{\underline{j+1},z} + P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{j+1,1} & -H_{\underline{j+1},z+1} + (*) & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{j+1,p_{\underline{j+1}}} & -P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} < 0$$

$$\Omega_{j+1,l} = A_{\underline{j+1},z+l-1} H_{\underline{j+1},z+l-1} - B_{\underline{j+1},z+l-1} F_{\underline{j+1},z+l-1}$$

**Sufficient LMI conditions exist**

**Extension: $\alpha$-sample variation**

# Conditions – on an example

- 2 subsystems
- periods: $p_1 = 2$, $p_2 = 2$

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))(A_{1i}\boldsymbol{x}(k) + B_{1i}\boldsymbol{u}(k)) \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))(A_{2i}\boldsymbol{x}(k) + B_{2i}\boldsymbol{u}(k)) \end{cases}$$

# Sufficient conditions

$$\begin{pmatrix} -H_{2,z} + (*) + P_{1,z} & (*) & (*) \\ A_{2,z}H_{2,z} - B_{2,z}F_{2,z} & -H_{2,z+1} + (*) & (*) \\ 0 & A_{2,z+1}H_{2,z+1} - B_{2,z+1}F_{2,z+1} & -P_{2,z+2} \end{pmatrix} < 0$$

$$\begin{pmatrix} -H_{1,z} + (*) + P_{2,z} & (*) & (*) \\ A_{1,z}H_{1,z} - B_{1,z}F_{1,z} & -H_{1,z+1} + (*) & (*) \\ 0 & A_{1,z+1}H_{1,z+1} - B_{1,z+1}F_{1,z+1} & -P_{1,z+2} \end{pmatrix} < 0$$

# Example

$$A_{11} = \begin{pmatrix} 1.5 & 10 \\ 0 & 0.5 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0.5 & 10 \\ 0 & 0 \end{pmatrix}$$

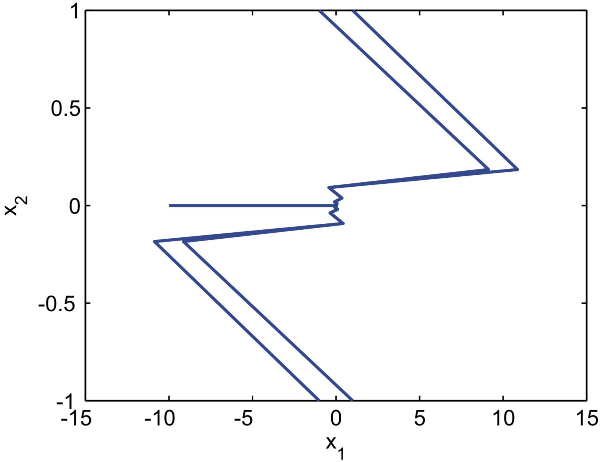
$$A_{21} = \begin{pmatrix} 1+a & 1 \\ 0 & 0.5 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 10 \\ 0 & 0.5 \end{pmatrix}$$

$$B_{11} = B_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad B_{21} = B_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- $a$ real-valued parameter
- first subsystem not controllable
- $A_{11}$ unstable

# Results

*a* can be increased to 1500



# Conclusions

- Periodic nonlinear discrete-time models
- Switching Lyapunov functions
- Analysis and design conditions
- Able to handle unstable/unobservable/uncontrollable local models

- Reducing conservativeness
- Generalization

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