Fuzzy modeling and control of HIV infection

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***Abstract*—In this paper we use a fuzzy model to represent the nonlinear dynamics of the Human Immunodeficiency Virus. To estimate the unmeasured variables we use an observer and we design a fuzzy controller to stabilize the system.**

I. Introduction

Human Immunodeficiency Virus (HIV) is a pathogen that infects T-helper cells of the immune system and can cause AIDS (Acquired Immune Deficiency Syndrome). In recent years, two major types of anti-HIV drugs were discovered: 1) the reverse transcriptase inhibitors (RTI) and 2) protease inhibitors (PI). These two treatments have good results, but the virus may suffer mutations and sometimes there were side effects. Using a mathematical model it is easier to study how the virus evolves in the body. Wodarz and Nowak [8] created a mathematical model that represents the HIV dynamics and use the model to observe how the infection progresses in the body for a specific treatment. Shanonn Kubiak et al [1] use a mathematical model that represents the HIV dynamics while testing a new type of cure called *Highly Active Anti-Retrovial Therapy (HAART)*. This cure had good results but it can also cause side effects.

Existing results are based on a linearized model, that due to the linearization and simplification, it is not accurate. In this paper, we propose to use the full nonlinear model proposed in [8]. In order to ease both the analysis and design, we use an exact fuzzy representation of this model.

The paper is structured as follows: Section II presents the HIV virus mathematical model, Section III explains Takagi-Sugeno (TS) fuzzy modeling, Section IV is the fuzzy modeling for HIV dynamics and Section V concludes the paper.

II. HIV mathematical model

The model for HIV infection dynamic created by *Wodarz and Nowak* [8] studies how the HIV affects the immune system during the natural course of infection. The model suggest that the cytotoxic T lymphocyte (CTL) has a memory which helps them to control the virus. Wodarz and Novak define CTL memory as long-term persistence of CTL precursors in the absence of antigen. The CTL memory can be enabled by the antiviral drug therapy. For the chronically infected patients, the CTL memory can be re-established by including in the treatment, either deliberate drug holidays or antigenic boosts of the immune system. Whether such treatment regimes would lead to long-term immunologic control deserves investigation under carefully controlled conditions.

In [1] the authors have changed the original model by adding an additional state v, which represents the viral load. The modified model is a coupled system of five ordinary differential equations with twelve parameters as follows

$$\begin{aligned}
\dot{\overline{x}}(t) &= \lambda - d\overline{x}(t) - \beta[1 - fu(t)]\overline{x}(t)\overline{v}(t) \\
\dot{\overline{y}}(t) &= \beta[1 - fu(t)]\overline{x}(t)\overline{v}(t) - a\overline{y}(t) - p\overline{y}(t)\overline{z}(t) \\
\dot{\overline{w}}(t) &= c\overline{x}(t)\overline{y}(t)\overline{w}(t) - cq\overline{y}(t)\overline{w}(t) - b\overline{w}(t) \\
\dot{\overline{z}}(t) &= cq\overline{y}(t)\overline{w}(t) - h\overline{z}(t) \\
\dot{\overline{v}}(t) &= k\overline{y}(t) - \mu\overline{v}(t)
\end{aligned} \tag{1}$$

where: $\overline{x}$ denotes the uninfected T helper cells, $\overline{y}$–infected T helper cells, $\overline{w}$–immune precursors CTL, $\overline{z}$–immune effectors CTL, $\overline{v}$–free virus.

In the model (1) [1] assume that the virus instantaneously approaches T-cells with no time lag for diffusion. In reality such a delay exists. $\overline{v}$ represents only the virions that can infect uninfected cells, represented by $\overline{x}$. The immune precursors ($\overline{w}$) are stimulated by infected cells, not by the virus. $u(t)$ represents the treatment in this model. Values of $u(t)$ range from 0 to 1, with 0 representing no treatment and 1 representing full treatment.

In this paper, we consider the parameter values presented in Table 1, adopted from [1].

Symbol	Value	Meaning
λ	1	The target cell production rate
d	0.1	Natural death rate of target cells
β	0.02	Rate of viral replication
a	0.2	Natural death rate of infected cells
p	1	Death rate of infected cells due to immune response
c	0.027	CTL activation rate
q	0.5	Growth rate of CTL effectors and precursors due to infected cells
b	0.001	Natural death rate of CTL precursors
h	0.1	Natural death rate of CTL effectors
k	25	Growth rate of virions due to infected cells
μ	1	Natural death rate of virions
f	0.75	Treatment efficacy factor

Table 1. Parameter values

For the parameters in Table 1, there are two equilibria which correspond to the success or failure of the immune system to control the infection. The equilibria values are presented in Table 2.

State	1st equilibrium	2nd equilibrium
$\overline{x}$	0.4	9.8
$\overline{y}$	4.8	0.004
$\overline{w}$	0	8751
$\overline{z}$	0	4.7
$\overline{v}$	120	0.1

Table 2. Equilibria values

Our goal is to stabilize the system (1) in the first equilibrium values, which corresponds to a successful treatment. To do this, we derive a Takagi-Sugeno fuzzy model that exactly represents the system (1). The modeling process is described in the following sections.

III. Takagi-Sugeno Fuzzy Modeling

In this section we present fuzzy modeling, stability analysis, observer and controller design.

1) The fuzzy model: We need mathematical models that represent the behavior of a physical dynamic system. If we have nonlinear systems we can decompose them into sub-systems which are representative for the respective sub-domains. Fuzzy modeling is a technique to build a multiple-model of the process based on the input-output data or the original mathematical model of the system.

Takagi and Sugeno [5] proposed a fuzzy model based on IF-THEN rules, which represent the local input-output relations of a nonlinear system. The main feature of the dynamic TS fuzzy model is to express the local dynamics by a linear system model.

The IF-THEN rules are of the form [2]:
Model rule i:
If z_1 *is* Z_1^i *and ... and* z_p *is* Z_p^i *then* **y**$=A_i(z)$

where the vector $\mathbf{z}$ has p components, z_j, $j = 1, 2, ...p$, and stands for the vector of antecedent variables. These variables are also called *scheduling variables*, as their values determine the degree to which the rules are active. The sets Z_j^i, $j = 1, 2, ...p$, $i = 1, 2, ...m$, where m is the number of rules, are the antecedent fuzzy sets.

The values of scheduling variables z_j, $j = 1, 2, ...p$ belong to a fuzzy set Z_j^i, $j = 1, 2, ...p$, $i = 1, 2, ...m$, with a truth value given by the membership functions h_i, $i = 1, 2, ...m$. Each membership function is composed using several weighting functions, w_{ji}, $j = 1, 2, ...p$, $i = 1, 2$.

Assuming that each scheduling variable is bounded, we can determine min(z_j) and max(z_j), $j = 1, 2, ...p$, respectively. For the min and max we chose a weighting function. For min(z_j) chose w_{j1} and for max(z_j) chose w_{j2}, $j = 1, 2, ...p$. The scheduling variable z_j, $j = 1, 2, ...p$, can be represented using the weighting functions w_{j1} and w_{j2}, $j = 1, 2, ...p$, as follows:

$$z_j = w_{j1} \cdot \text{min}(z_j) + w_{j2} \cdot \text{max}(z_j) \tag{2}$$

The weighting functions, w_{j1} and w_{j2}, $j = 1, 2, ...p$, have the form

$$w_{j1} = \frac{\text{max}(z_j) - z_j}{\text{max}(z_j) - \text{min}(z_j)}, w_{j2} = \frac{z_j - \text{min}(z_j)}{\text{max}(z_j) - \text{min}(z_j)}$$

and

$$w_{j1}(z_j) + w_{j2}(z_j) = 1 \tag{3}$$

The output of a rule i depends on the scheduling variables.

In A_i, $i = 1, 2, ...m$, we have min(z_j) or max(z_j), $j = 1, 2, ...p$, depending on the rule. If in the rule we have w_{j1}, $j = 1, 2, ...p$, we use min(z_j) in A_i, for w_{j2} we use max(z_j) in A_i.

Example Consider the dynamic system

$$\dot{\mathbf{x}} = \begin{pmatrix} z_1 & a \\ z_2 & b \end{pmatrix} \mathbf{x}$$

where z_1 and z_2 are scheduling variables and $\mathbf{x} = [x_1, x_2]$. The system matrix is

$$A(\mathbf{z}) = \begin{pmatrix} z_1 & a \\ z_2 & b \end{pmatrix}$$

This will be represented by a TS model and $\mathbf{z} = [z_1, z_2]$. We need four weighting functions, each z_j, $j = 1, 2$ needs two, one for min(z_j) and one for max(z_j). So we chose the following weighting functions: $w_{11}(z_1)$, $w_{12}(z_1)$, $w_{21}(z_2)$, $w_{22}(z_2)$, where $w_{j1}(z_j)$ corresponds to min(z_j) and $w_{j2}(z_j)$ corresponds to max(z_j). The resulting TS model is:
Model rule 1: If z_1 is w_{11} and z_2 is w_{21} then $\dot{\mathbf{x}} = A_1\mathbf{x}$
Model rule 2: If z_1 is w_{12} and z_2 is w_{21} then $\dot{\mathbf{x}} = A_2\mathbf{x}$
Model rule 3: If z_1 is w_{11} and z_2 is w_{22} then $\dot{\mathbf{x}} = A_3\mathbf{x}$
Model rule 4: If z_1 is w_{12} and z_2 is w_{22} then $\dot{\mathbf{x}} = A_4\mathbf{x}$
where $\mathbf{x} = [x_1, x_2]$. The matrices A_i, $i = 1 \ldots 4$ in this case are

$$A_1(\mathbf{z}) = \begin{pmatrix} \text{min}(z_1) & a \\ \text{min}(z_2) & b \end{pmatrix}, A_2(\mathbf{z}) = \begin{pmatrix} \text{max}(z_1) & a \\ \text{min}(z_2) & b \end{pmatrix}$$

$$A_3(\mathbf{z}) = \begin{pmatrix} \text{min}(z_1) & a \\ \text{max}(z_2) & b \end{pmatrix}, A_4(\mathbf{z}) = \begin{pmatrix} \text{max}(z_1) & a \\ \text{max}(z_2) & b \end{pmatrix}$$

There are 4 membership functions, h_i, $i = 1 \ldots 4$:

$$\begin{aligned} h_1 &= w_{11}(z_1) \cdot w_{21}(z_2) \\ h_2 &= w_{11}(z_1) \cdot w_{22}(z_2) \\ h_3 &= w_{12}(z_1) \cdot w_{21}(z_2) \\ h_4 &= w_{12}(z_1) \cdot w_{22}(z_2) \end{aligned}$$

$\dot{\mathbf{x}}$ can be derived as [4]:

$$\dot{\mathbf{x}} = \sum_{i=1}^{4} h_i(\mathbf{z}) A_i(\mathbf{z}) \mathbf{x} \tag{4}$$

This method will be applied in Section IV on the HIV mathematical model.

2) Lyapunov stability: An important feature of dynamical systems is stability. One of the methods used to study the stability of nonlinear systems is the Lyapunov method.

There is no general procedure for finding Lyapunov functions for nonlinear systems. The Lyapunov function can be chosen to be quadratic, that is [2]

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x},\ P > 0,\ P = P^T$$

In the case of fuzzy systems, the system model is [2]:

$$\dot{x} = \sum_{i=1}^{m} h_i(\mathbf{z})A_i\mathbf{x}$$

The Lyapunov function will be:

$$\begin{aligned}
&V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}\\
&\dot{V}(\mathbf{x}) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} =\\
&= \sum_{i=1}^{m} h_i(\mathbf{z})(A_i\mathbf{x})^T P\mathbf{x} + \sum_{i=1}^{m} h_i(\mathbf{z})\mathbf{x}^T PA_i\mathbf{x} =\\
&= \sum_{i=1}^{m} h_i(\mathbf{z})\mathbf{x}^T(A_i^T P + PA_i)\mathbf{x}
\end{aligned}$$

In this case the system is stable if there exists $P = P^T > 0$ so that $A_i^T P + PA_i < 0,\ \forall i = 1\ldots m$.

3) The estimator: Applying a control low requires knowing the values of states. In practice it is not possible to measure all the system states. The solution for this problem are state observers. A state observer estimates the process states relying on the process mathematical model, using the input and the output of the process.

A TS fuzzy system has the form:

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \sum_{i=1}^{m} h_i(\mathbf{z})(A_i\mathbf{x}(t) + B_i\mathbf{u}(t))\\
\mathbf{y}(t) &= \sum_{i=1}^{m} h_i(\mathbf{z})C_i\mathbf{x}(t)
\end{aligned}$$

where h_i are the membership functions, B_i are the input matrices, C_i are the output matrices, $i = 1...m$, $\mathbf{x} = [x_1, x_2, ..., x_n]$–vector of systems states, $\mathbf{u}(t)$–input, $\mathbf{y} = [y_1, y_2, ..., y_l]$–output of the system, A_i–state matrix of rule i, $i = 1...m$, m–number of rules. Assuming that the scheduling variables are known, the general form of a fuzzy estimator is:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= \sum_{i=1}^{m} h_i(\mathbf{z})(A_i\hat{\mathbf{x}}(t) + B_i\mathbf{u}(t) + L_i(\mathbf{y} - \hat{\mathbf{y}}))\\
\hat{\mathbf{y}}(t) &= \sum_{i=1}^{m} h_i(\mathbf{z})C_i\hat{\mathbf{x}}(t)
\end{aligned}$$

The error between the original system and the estimator is

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \sum_{i=1}^{m} h_i(\mathbf{z}) \sum_{j=1}^{m} h_j(\mathbf{z})(A_i - L_iC_j)\mathbf{e}$$

where $\mathbf{e} = [e_1, e_2, ..., e_n]$–vector of errors, n–number of states. To verify the stability of the error dynamics, one can use the Lyapunov function V, of the form:

$$\begin{aligned}
&V = \mathbf{e}^T P\mathbf{e}, P > 0\\
&\dot{V} = \sum_{i=1}^{m} h_i(\mathbf{z}) \sum_{j=1}^{m} h_j(\mathbf{z})\mathbf{e}^T[P(A_i - L_iC_j) +\\
&+ (A_i - L_iC_j)^T P]\mathbf{e}
\end{aligned}$$

The error system is stable if $\dot{V} < 0$ which is satisfied if $P(A_i - L_iC_j) + (A_i - L_iC_j)^T P < 0,\ i = 1...m,\ j = 1...m$. If we denote $M_i = PL_i$ then the stability condition become

$$PA_i - M_iC_j + (PA_i - M_iC_j)^T < 0, i = 1...m, j = 1...m \quad (5)$$

which is an LMI (linear matrix inequality) that can be easily solved.

If we denote $G_{ij} = PA_i - M_iC_j + (PA_i - M_iC_j)^T$, $i = 1...m,\ j = 1...m$ then we have the following stability conditions [6]:

$$\begin{aligned}
&G_{ii} < 0\\
&\frac{2}{m-1}G_{ii} + G_{ij} + G_{ji} < 0
\end{aligned}$$

where m is the number of rules and $G_{ii} = PA_i - M_iC_i + (PA_i - M_iC_i)^T$.

4) Controller: The PDC (parallel distributed compensator) controller used in fuzzy control has the following form

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \sum_{i=1}^{m} h_i(\mathbf{z})(A_i\mathbf{x}(t) + B_i\mathbf{u}(t))\\
\mathbf{u}(t) &= -\sum_{j=1}^{m} h_j(\mathbf{z})F_j\mathbf{x}(t)
\end{aligned}$$

where F_i represent the local feedback gains, h_i–membership functions, A_i–state matrix of fuzzy rule i, $i = 1...m$, B_i–input matrices, $\mathbf{u}(t)$– control input.

To verify the stability of the controller, one can use the Lyapunov function:

$$\begin{aligned}
&V = \mathbf{x}^T P\mathbf{x}, P > 0\\
&\dot{V} = \mathbf{x}^T \sum_{i=1}^{m} h_i(\mathbf{z}) \sum_{j=1}^{m} h_j(\mathbf{z})(P(A_i - B_iF_j) +\\
&+ (A_i - B_iF_j)^T P)\mathbf{x} < 0
\end{aligned}$$

The controller is stable if $P(A_i - B_iF_j) + (A_i - B_iF_j)^T P < 0,\ i = 1...m,\ j = 1...m$. Denoting $X = P^{-1}$ the relation becomes

$$(A_i - B_iF_j)X + X(A_i - B_iF_j)^T < 0,\ i = 1...m,\ j = 1...m$$

If we denote $M_j = F_jX,\ j = 1...m$ then we have

$$A_iX - B_iM_j + XA_i^T - M_j^T B_i^T < 0,\ i = 1...m,\ j = 1...m, \quad (6)$$

which is a LMI. Let $G_{ij} = A_iX - B_iM_j$, $i = 1...m$, $j = 1...m$ and the stability conditions are [7]

$$G_{ii}^T + G_{ii} < 0 \quad (7)$$

$$(\frac{G_{ij}+G_{ji}}{2})^T+(\frac{G_{ij}+G_{ji}}{2})<0 \quad (8)$$

where $G_{ii}=A_iX-B_iM_i$. If we have $B_1=B_2=B_3=...=B_m$ then we need to find $P>0$ that satisfies only the inequality (7).

IV. Fuzzy modeling of HIV dynamics

In this section we derive the equivalent fuzzy model for the system described by (1). Using the 1st equilibrium values from Table 2, we modify the system so that it has an equilibrium in zero.

To move the system equilibrium in zero we make the following replacement: $\overline{x}=x+0.4$, $\overline{y}=y+4.8$, $\overline{w}=w+0$, $\overline{z}=z+0$ and $\overline{v}=v+120$. Assuming a constant input, the new system is:

$$\begin{aligned}
\dot{x}&=(-2.5-0.02v)x-0.008v\\
\dot{y}&=(2.4+0.02v)x+0.008v+(-0.2-z)y-4.8z\\
\dot{w}&=(0.027y+0.1296)xw+(-0.0027y-0.014)w\\
\dot{z}&=(0.0135y+0.0648)w-0.1z\\
\dot{v}&=25y-v
\end{aligned} \quad (9)$$

As we can see in (9), there are six nonlinear terms, so we need to chose six scheduling variables: z_1, z_2, z_3, z_4, z_5, z_6. The nonlinearities associated to the scheduling variables are

$$\begin{aligned}
z_1&=-2.5-0.02v,\\
z_2&=2.4+0.02v,\\
z_3&=0.027yw+0.1296w,\\
z_4&=-0.2-z,\\
z_5&=-0.0027y-0.014,\\
z_6&=0.0135y+0.0648.
\end{aligned}$$

The matrix of system (9) is

$$A(\mathbf{x})=\begin{pmatrix} z_1 & 0 & 0 & 0 & -0.008\\ z_2 & z_4 & 0 & -4.8 & 0.008\\ z_3 & 0 & z_5 & 0 & 0\\ 0 & 0 & z_6 & -0.1 & 0\\ 0 & 25 & 0 & 0 & -1 \end{pmatrix} \quad (10)$$

To define the weighting functions, we calculate the minimum and the maximum values of $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$, $z_5(t)$ and $z_6(t)$, which under the assumption $x,y,w,z,v\in[0,10]$ are:

min($z_1(t)$)=-2.7	max($z_1(t)$)=-2.5
min($z_2(t)$)=2.4	max($z_2(t)$)=2.6
min($z_3(t)$)=0	max($z_3(t)$)=3.996
min($z_4(t)$)=-10.2	max($z_4(t)$)=-0.02
min($z_5(t)$)=-0.041	max($z_5(t)$)=-0.014
min($z_6(t)$)=0.0648	max($z_6(t)$)=0.1998

Therefore w_{11}, w_{12} ,w_{21}, w_{22}, w_{31}, w_{32}, w_{41}, w_{42}, w_{51}, w_{52}, w_{61}, w_{62} can be represented using the scheduling variables z_1, z_2, z_3, z_4, z_5 z_6, as follows:

$$\begin{aligned}
&w_{11}(z_1)=0.1v, w_{12}(z_1)=1-0.1v\\
&w_{21}(z_2)=1-0.1v, w_{22}(z_2)=0.1v\\
&w_{31}(z_3)=1-0.0068yw-0.0324w\\
&w_{32}(z_3)=0.0068yw+0.0324w\\
&w_{41}(z_4)=\frac{z}{10.18}, w_{42}(z_4)=1-\frac{z}{10.18}\\
&w_{51}(z_5)=0.1y, w_{52}(z_5)=1-0.1y\\
&w_{61}(z_6)=1-0.1y, w_{62}(z_6)=0.1y
\end{aligned} \quad (11)$$

and we have

$$w_{j1}(z_j)+w_{j2}(z_j)=1,\ j=1...6$$

The scheduling variables represented using the weighting functions are:

$$\begin{aligned}
z_1&=w_{12}\cdot(-2.7)+w_{12}\cdot(-2.5)\\
z_2&=w_{21}\cdot(2.4)+w_{22}\cdot(2.6)\\
z_3&=w_{31}\cdot(0)+w_{32}\cdot(3.996)\\
z_4&=w_{41}\cdot(-10.2)+w_{42}\cdot(-0.02)\\
z_5&=w_{51}\cdot(-0.041)+w_{52}\cdot(-0.014)\\
z_6&=w_{61}\cdot(0.0648)+w_{62}\cdot(0.1998)
\end{aligned}$$

The model rules in this case are complex. The number of rules in the case of HIV virus system is 64, but since $w_{11}=w_{22}$, $w_{12}=w_{21}$, $w_{51}=w_{62}$ and $w_{52}=w_{61}$, we have only 16 rules. For instance, one of the rules is:

Model rule 1:
If $z_1(t)$ is w_{11} and $z_3(t)$ is w_{31} and $z_4(t)$ is w_{41} and $z_5(t)$ is w_{51} Then $\dot{\mathbf{x}}=A_1\mathbf{x}$
where

$$A_1=\begin{pmatrix} -2.7 & 0 & 0 & 0 & -0.008\\ 2.5 & -10.2 & 0 & -4.8 & 0.008\\ 0 & 0 & -0.041 & 0 & 0\\ 0 & 0 & 0.1998 & -0.1 & 0\\ 0 & 25 & 0 & 0 & -1 \end{pmatrix}$$

Each model rule is a combination of $w_{1k}, w_{3k}, w_{4k}, w_{5k}$ where $k=1,2$. If in the rule we have w_{j1}, for each j we will use in matrix (10) the minimum value or maximum of z_j, at the proper position of z_j, $j=1...6$. If in the rule we have w_{j2}, for each j we will use in matrix (10) the maximum value of z_j, at the proper position of z_j, $j=1...6$.

For instance, the rule corresponding to A_1 is the following combination of weighting functions: w_{11}, w_{31}, w_{41}, w_{51}, and the matrix A_1, in terms of min(z_j) and max(z_j), $j=1...6$ is

$$A_1=\begin{pmatrix} \min(z_1) & 0 & 0 & 0 & -0.008\\ \max(z_2) & \min(z_4) & 0 & -4.8 & 0.008\\ \min(z_3) & 0 & \min(z_5) & 0 & 0\\ 0 & 0 & \max(z_6) & -0.1 & 0\\ 0 & 25 & 0 & 0 & -1 \end{pmatrix}$$

Now, $\dot{\mathbf{x}}$ can be derived as:

$$\dot{\mathbf{x}} = h_1(\mathbf{z}(t))A_1\mathbf{x}(t) + h_2(\mathbf{z}(t))A_2\mathbf{x}(t) + \\ + h_3(\mathbf{z}(t))A_3\mathbf{x}(t) + \ldots + h_{16}(\mathbf{z}(t))A_{16}\mathbf{x}(t)$$

where $\mathbf{z} = [z_1, z_3, z_4, z_5]$ and

$$h_1(\mathbf{z}(t)) = w_{11}(z_1(t)) \cdot w_{31}(z_3(t)) \cdot w_{41}(z_4(t)) \cdot w_{51}(z_5(t))$$
$$h_2(\mathbf{z}(t)) = w_{11}(z_1(t)) \cdot w_{31}(z_3(t)) \cdot w_{41}(z_4(t)) \cdot w_{52}(z_5(t))$$
$$h_3(\mathbf{z}(t)) = w_{11}(z_1(t)) \cdot w_{31}(z_3(t)) \cdot w_{42}(z_4(t)) \cdot w_{51}(z_5(t))$$
$$\ldots$$
$$h_{16}(\mathbf{z}(t)) = w_{12}(z_1(t)) \cdot w_{32}(z_3(t)) \cdot w_{42}(z_4(t)) \cdot w_{52}(z_5(t))$$

The dynamic system is the same as the original (9).

The evolution of the states is presented in Fig. 1. Since the considered equilibrium point, *(0.4;4.8;0;0;120)*, is unstable, we need to calculate a controller. In the following sections we will present the observer and the controller design.

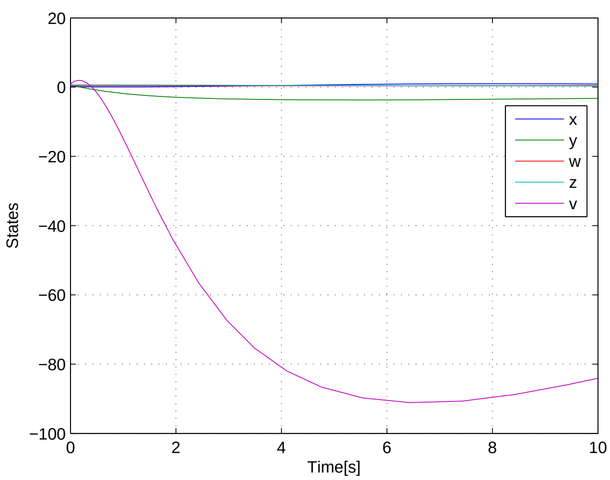

Fig. 1. Evolution of states

A. The estimator

In order to know all the states and to use them for control, an estimator is needed. The estimator equations have been given in Section III.

For this estimator we consider that there are three states that can be measured: x, y and v. The output matrix C will have the form:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Because we have only a matrix C, to verify the stability is sufficient to use only the criterion

$$G_{ii} < 0$$

where $G_{ii} = PA_i - M_iC + (PA_i - M_iC)^T$, $i = 1...16$. To design a stable estimator, like it is presented in Section III, we need to calculate a matrix P which is positive definite. Solving the LMI (5) using the Sedumi solver of the Yalmip [3] toolbox in Matlab we obtain:

$$P = \begin{pmatrix} 0.0488 & -0.0001 & 0.001 & -0.0004 & 0 \\ -0.0001 & 0.0564 & 0.0139 & 0.0162 & 0.0085 \\ 0.001 & 0.0139 & 0.53 & -0.2983 & 0.0036 \\ -0.0004 & 0.0162 & -0.2983 & 0.3159 & 0.0032 \\ 0 & 0.0085 & 0.0036 & 0.0032 & 0.0488 \end{pmatrix}$$

and 16 observer gains are obtained. For instance, the first gain is

$$L_1 = \begin{pmatrix} -0.4939 & 0.0033 & 0.0016 \\ 2.6148 & -7.2330 & -0.0373 \\ -0.0065 & -1.3604 & -0.2081 \\ -0.0032 & -2.2936 & -0.3466 \\ -0.0097 & 24.5733 & 1.2521 \end{pmatrix} \quad (12)$$

Fig. 2 presents the estimated values of the system states w and z. For the simulation, the initial conditions were $\mathbf{x} = [0.2,\ 0.6,\ 0.4,\ 0.7,\ 0.9]^T$, while the estimated initial states were $\hat{\mathbf{x}}_0 = [0.1,\ 0.5,\ 0,\ 0.3,\ 0.7]^T$. The error between the estimated values and the system states is presented in Fig. 3.

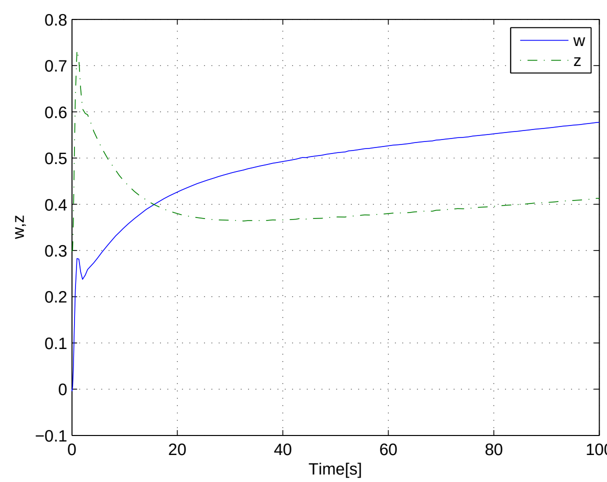

Fig. 2. Estimated states

Like we see in Fig. 3 at the beginning of estimation is a large error, but in time that error converges to zero.

B. The controller

The form of the controller is given in Section III. For this model we calculate a controller that stabilizes the system.

To test the controller we used the following input matrix B

$$B = (0\ 0\ 1\ 0\ 0)^T$$

Because we have only one B to design the controller is enough to use

$$G_{ii}^T + G_{ii} < 0 \quad (13)$$

where P is a positive define matrix and $G_{ii} = A_iX - BM_i$, $i = 1...16$. Solving the LMI (6) using the Sedumi solver of

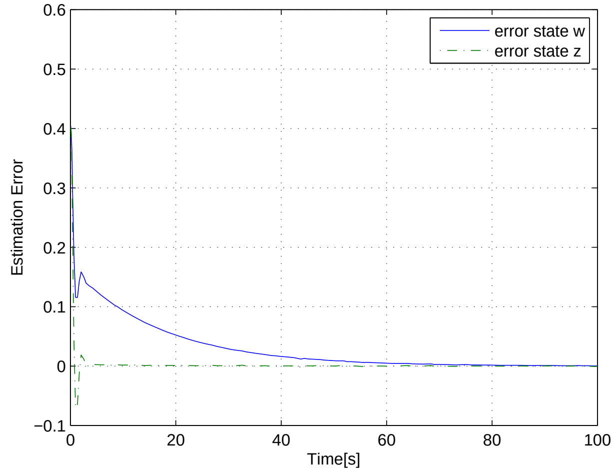

Fig. 3. Estimation error of states w and z

the Yalmip [3] toolbox in Matlab we obtain $X^{-1} = P$:

$$P = \begin{pmatrix} 0.4174 & 0.5191 & -0.0342 & -1.4161 & 0.0303 \\ 0.5191 & 0.7112 & -0.0445 & -1.7951 & 0.0364 \\ -0.0342 & -0.0445 & 0.003 & 0.1194 & -0.0024 \\ -1.4161 & -1.7951 & 0.1194 & 4.9286 & -0.1041 \\ 0.0303 & 0.0364 & -0.0024 & -0.1041 & 0.0028 \end{pmatrix} \quad (14)$$

The local feedback gains F_i, $i = 1...16$ have been calculated using $F_i = M_i X^{-1}$, $X = P^{-1}$. For example, the first gain is

$$F_1 = [\ -2.4513 \quad -3.1574 \quad 0.208 \quad 8.5414 \quad -0.1766\ ]$$

The evolution of the closed-loop states is presented in Fig. 4. As can be seen, the states of the system are stabilized.

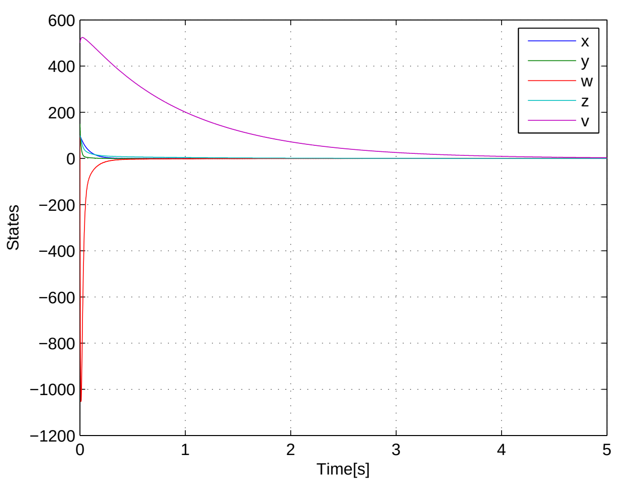

Fig. 4. Evolution of the closed-loop states

V. Conclusion

The model used in this work is nonlinear but using the fuzzy modeling we can work with linear sub-systems. The controller stabilizes the nonlinear system, like we can see in Fig. 4. In our future research, we will investigate a more realistic model and the inclusion of the performance indices in the design.

Acknowledgment

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