Output feedback control for T-S discrete-time nonlinear descriptor models*

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***Abstract*— This paper presents a static output feedback controller design for discrete-time nonlinear descriptor models. The conditions are given in terms of linear matrix inequalities (LMIs). The approach is based on the Takagi-Sugeno (T-S) representation of the nonlinear system and Finsler's Lemma. The proposed method exploits the discrete-time nature of the T-S model by the use of delayed Lyapunov functions, which provide more degrees of freedom without increasing the number of LMIs. It is also extended for robust control. The benefits of the proposed approaches are illustrated via numerical examples.**

I. Introduction

Designing controllers for nonlinear systems is often difficult to perform; in order to overcome the difficulties; many approaches exist in the literature. For instance, the sector nonlinearity approach allows rewriting a certain class of nonlinear models [1] as a collection of local linear models blended together by nonlinear membership functions (MFs) [2]. This representation is the so-called Takagi-Sugeno (T-S) model [3]. One of the advantages of expressing the nonlinear model as a T-S one is that the latter allows employing the direct Lyapunov method and obtaining stability/design conditions in terms of linear matrix inequalities (LMIs), which can be solved via convex optimization techniques [4]. Classically, the so-called Parallel Distributed Compensator (PDC) [2] has been used for the controller design. The PDC is a nonlinear controller since it is a convex combination of local linear gains using the nonlinear MFs of the T-S model. Once the PDC controller is designed, it can be directly applied to the nonlinear plant [5], [6].

However, the TS-LMI framework has some disadvantages. One of the sources of conservativeness is the use of quadratic Lyapunov functions, i.e., a common Lyapunov matrix must be found for all the linear local models [2]. For discrete-time T-S models, the use of non-quadratic Lyapunov functions has reduced conservativeness [7]–[9]; recently, delayed non-PDC controllers together with delayed non-quadratic Lyapunov functions have been developed [10], [11].

Using the sector nonlinearity approach, another disadvantage appears when the original nonlinear model has several p nonlinear terms, since the number of linear models is $r = 2^p$. Based on a nonlinear descriptor model [12], in [13], the T-S descriptor model was introduced. There are several works with applications concerning this T-S representation [14]–[16]. The T-S descriptor system is helpful because it reduces the number of LMI constraints and keeps the natural descriptor form of the nonlinear system.

Unfortunately, when using state feedback techniques, it is necessary to have available all the states of the plant; in general this is not possible [6], [17]–[19]. Hence, an output feedback controller (OFC) has to be implemented [19], [20]. Generally, two different OFCs can be designed: 1) static OFC (SOFC) [21]–[25]; 2) dynamic OFC (DOFC) [20], [26], [27]. The design of a DOFC increases the dimension of the closed-loop system. In the case of SOFC, several results exist: the first results provided BMI conditions [23], [24]; later on, results based on LMI and equality conditions have been developed [21], [25]; the main disadvantage of these approaches is that they consider a common Lyapunov matrix and common output matrices (without uncertainties or without nonlinearities in the output matrix). Recently, strict LMI conditions have been proposed in [28]; this result also provides solution for problems with multiple output matrices and gives conditions for the robust control problem.

The main objective of the present paper is to develop SOFC for T-S descriptor models via LMI conditions, where multiple output matrices can be taken into account. To that end, we use: 1) T-S descriptor models which are more general than standard T-S models; 2) the well-known Finsler's Lemma [29] to remove the link between the controller gains and the Lyapunov matrices [10], [28]; 3) delayed non-PDC controllers and delayed non-quadratic Lyapunov functions in order to reduce the conservativeness [10].

The paper is divided as follows: Section II introduces the T-S descriptor model, notation, properties and lemmas; Section III presents the main results on the SOFC for discrete-time T-S descriptor models and extends the new results for T-S descriptor models with uncertainties; Section IV illustrates the advantages of the approaches via numerical examples; Section V concludes the paper.

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II. NOTATIONS AND PROBLEM STATEMENT

Consider the following nonlinear descriptor model:

$$\begin{aligned} E(x(\kappa))x(\kappa+1) &= A(x(\kappa))x(\kappa)+B(x(\kappa))u(\kappa) \\ y(\kappa) &= C(x(\kappa))x(\kappa), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input vector, $y \in \mathbb{R}^o$ is the output vector, κ is the current sample. Matrices $A(x(\kappa))$, $B(x(\kappa))$, $C(x(\kappa))$, and $E(x(\kappa))$ are assumed to be smooth in a compact set Ω_x of the state space including the origin. In addition, in this work the matrix $E(x(\kappa))$ is assumed to be regular.

Via the sector nonlinearity approach [1] the p nonlinear terms in matrices $(A(x), B(x), C(x))$ and the p_e nonlinearities in matrix $E(x)$ can be grouped via MFs [2]. These MFs hold the convex-sum property in the compact set Ω_x, i.e., $h_i(z(\kappa)) \geq 0$, $i \in \{1,\ldots,2^p\}$, $\sum_{i=1}^{r} h_i(z(\kappa)) = 1$, $v_k(z(\kappa)) \geq 0$, $k \in \{1,\ldots,2^{p_e}\}$, $\sum_{k=1}^{r_e} v_k(z(\kappa)) = 1$; they depend on the premise variables grouped in the vector $z(\kappa)$ which, in this work, depends on measured variables. Moreover, $x_{\kappa+}$ and x_κ stand for $x(\kappa+1)$ and $x(\kappa)$ respectively.

The following discrete-time T-S descriptor model is an exact representation of (1) in the compact set Ω_x:

$$\begin{aligned} \sum_{k=1}^{r_e} v_k(z(\kappa)) E_k x_{\kappa+} &= \sum_{i=1}^{r} h_i(z(\kappa))(A_i x_\kappa + B_i u_\kappa) \\ y_\kappa &= \sum_{i=1}^{r} h_i(z(\kappa)) C_i x_\kappa, \end{aligned} \tag{2}$$

where matrices A_i, B_i, and C_i, $i \in \{1,\ldots,r\}$ represent the *i*-th linear model in the right-hand side of the T-S descriptor model and matrices E_k, $k \in \{1,\ldots,r_e\}$ represent the *k*-th linear model in the left-hand side of (2). It is important to stress that the T-S model (2) is strictly equivalent to (1) in the pre-specified compact set Ω_x.

The following shorthand notation is used throughout the paper: $\Upsilon_h = \sum_{i=1}^{r} h_i(z(\kappa))\Upsilon_i$, $\Upsilon_h^{-1} = \left(\sum_{i=1}^{r} h_i(z(\kappa))\Upsilon_i\right)^{-1}$, $\Upsilon_{h^-} = \sum_{l=1}^{r} h_l(z(\kappa-1))\Upsilon_l$, $\Upsilon_{hh} = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(\kappa)) h_j(z(\kappa))\Upsilon_{ij}$. An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side. Arguments will be omitted when their meaning is clear.

In order to obtain LMI conditions, the following relaxation scheme will be employed due to its good compromise between effectiveness and computational complexity:

Lemma 1 [30] (Relaxation Lemma): Let Υ_{ij}^k be matrices of appropriate dimensions. Consider the following inequality

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} h_i(z(\kappa)) h_j(z(\kappa)) v_k(z(\kappa)) \Upsilon_{ij}^k < 0.$$

Then $\Upsilon_{hhv} < 0$ holds if

$$\Upsilon_{ii}^k < 0,\ i \in \{1,\ldots,r\},\ k \in \{1,\ldots,r_e\},$$

$$\frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0,\ i,j \in \{1,\ldots,r\},\ k \in \{1,\ldots,r_e\},\ i \neq j.$$

The approaches are based on the following lemma, since it allows separating the control law and the Lyapunov matrix.

Finsler's Lemma [29]: Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $rank(R) < n$; the following expressions are equivalent

a) $x^T Q x < 0$, $\forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\}$.

b) $\exists M \in \mathbb{R}^{n\times m} : Q + MR + R^T M^T < 0$.

Property 1. Let $Q = Q^T > 0$, R, and M be matrices of appropriate sizes. The following expression holds:

$$R^T M + M^T R \leq R^T Q R + M^T Q^{-1} M.$$

The following example illustrates the advantage of keeping the descriptor form instead of computing the standard representation, i.e.,

$$x_{\kappa+} = A(x)x_\kappa + B(x)u_\kappa. \tag{3}$$

Example 1. Consider a nonlinear discrete-time descriptor system as (1) with $E(x) = \begin{bmatrix} 0.9 & 0.1-0.1x_2 \\ -0.4-0.15x_2 & 1.1 \end{bmatrix}$, $A(x) = \begin{bmatrix} -0.5 & 0.8-0.1x_2^2 \\ -1.2 & 0.5 \end{bmatrix}$, $C(x) = \begin{bmatrix} 0 \\ 1.3-0.15x_2^2 \end{bmatrix}^T$, and $B(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Consider the compact set $\Omega_x \in \{x : x_1 \in \mathbb{R}, |x_2| \leq 2\}$; inside this compact set the matrix $E(x)$ is nonsingular. A T-S descriptor model (2) has $r_e = 2$ due to x_2 (left-hand side) and $r = 2$ due to x_2^2 (right-hand side). In order to apply the SOFC conditions in [28], it is necessary to write (1) in form (3), and to compute $(E(x))^{-1}$, thus it writes $x_{\kappa+} = (E(x))^{-1}(A(x)x_\kappa + B(x)u_\kappa)$ with

$$(E(x))^{-1} = \frac{-10}{-206+5x_2+3x_2^2}\begin{bmatrix} 22 & -2+x_2 \\ 8+3x_2 & 18 \end{bmatrix}.$$

Using the sector nonlinearity approach in Ω_x, four different nonlinearities must be take into account, i.e.,

$$x_2^2,\quad (-10)/(-206+5x_2+3x_2^2),$$
$$\frac{-10(8+3x_2)}{-206+5x_2+3x_2^2},\quad \frac{-10(-2+x_2)}{-206+5x_2+3x_2^2},$$

which leads to $r=2^4=16$ linear models. This example illustrates that T-S descriptor model produces less rules (linear models); this example is continued in Section 4. ◊

The next section presents a new SOFC for the stabilization of T-S descriptor systems.

III. Main Results

A. Stabilization via SOFC

In [21], [25], [28] the following PDC control law is used:

$$u_\kappa = K_h y_\kappa . \tag{4}$$

In this work, the SOFC design is done by the use of a delayed non-PDC control law [10], [11]:

$$u_\kappa = \left(H_{hhh^-v}\right)^{-1} K_{hh^-v} y_\kappa \ , \tag{5}$$

where $H_{hhh^-v} \in \mathbb{R}^{m\times m}$, $K_{hh^-v} \in \mathbb{R}^{m\times o}$ are the gain matrices to be calculated. They have a convex structure, for instance:

$$K_{hh^-v} = \sum_{j=1}^{r}\sum_{l=1}^{r}\sum_{k=1}^{r_e} h_j\left(z(\kappa)\right) h_l\left(z(\kappa-1)\right) v_k\left(z(\kappa)\right) K_{jl}^{k} .$$

The control law (5) includes, via the MFs, all the nonlinear terms in both sides of the T-S descriptor system, i.e., it is a nonlinear control law.

A useful rewriting of the T-S descriptor model (2) and the delayed non-PDC control law (5) gives

$$\begin{bmatrix} A_h & -E_v & B_h \\ \left(H_{hhh^-v}\right)^{-1} K_{hh^-v} C_h & 0_{m\times n} & -I_m \end{bmatrix} \begin{bmatrix} x_\kappa \\ x_{\kappa+} \\ u_\kappa \end{bmatrix} = 0 . \tag{6}$$

Using a delayed Lyapunov function candidate [10]:

$$V(x_\kappa) = x_\kappa^T \left(\sum_{l=1}^{r} h_l\left(z(\kappa-1)P_l\right)\right) x_\kappa = x_\kappa^T P_{h^-} x_\kappa > 0 , \tag{7}$$

its variation is

$$\Delta V(x_\kappa) = x_{\kappa+}^T P_h x_{\kappa+} - x_\kappa^T P_{h^-} x_\kappa < 0 , \tag{8}$$

where $P_j = (P_j)^T > 0$, $P_j \in \mathbb{R}^{n\times n}$, $j \in \{1,\dots,r\}$. One can add $u_\kappa^T 0 u_\kappa$ to (8) without altering the inequality. Thus (8) writes

$$\Delta V(x_\kappa) = \begin{bmatrix} x_\kappa \\ x_{\kappa+} \\ u_\kappa \end{bmatrix}^T \begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_\kappa \\ x_{\kappa+} \\ u_\kappa \end{bmatrix} < 0 . \tag{9}$$

Via Finsler's lemma the inequality (9) under constraint (6) gives

$$\begin{bmatrix} -P_{h^-} & 0 & 0 \\ 0 & P_h & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix} \begin{bmatrix} A_h & -E_v & B_h \\ \left(H_{hhh^-v}\right)^{-1} K_{hh^-v} C_h & 0_{m\times n} & -I_m \end{bmatrix} + (*) < 0, \tag{10}$$

where $\mathcal{F},\ \mathcal{G} \in \mathbb{R}^{n\times(n+m)}$, $\mathcal{H} \in \mathbb{R}^{m\times(n+m)}$. The following result can be stated:

Theorem 1: The T-S descriptor system (2) together with the control law (5) is asymptotically stable if there exist matrices $P_h = (P_h)^T > 0$, G_{hh^-} H_{hhh^-v}, and K_{hh^-v} such that the following inequality holds

$$\Upsilon_{hhh^-v} = \begin{bmatrix} -P_{h^-} & (*) & (*) \\ G_{hh^-}A_h + \eta K_{hh^-v} C_h & \Gamma^{(2,2)} & (*) \\ K_{hh^-v} C_h & \Gamma^{(3,2)} & -H_{hhh^-v} + (*) \end{bmatrix} < 0, \tag{11}$$

where $\Gamma^{(2,2)} = -G_{hh^-}E_v - E_v^T G_{hh^-}^T + P_h$ and $\Gamma^{(3,2)} = \left(G_{hh^-}B_h - \eta H_{hhh^-v}\right)^T$.

Proof: Recall (10). Select

$$\begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix} = \begin{bmatrix} 0_n & 0_{n\times m} \\ G_{hh^-} & \eta H_{hhh^-v} \\ 0_{m\times n} & H_{hhh^-v} \end{bmatrix}, \tag{12}$$

where $G_{hh^-} \in \mathbb{R}^{n\times n}$ and $H_{hhh^-v} \in \mathbb{R}^{m\times m}$ are decision variables. Moreover, $\eta \in \mathbb{R}^{n\times m}$ is an arbitrary matrix to be chosen a priori, i.e., it is not a decision variable. Thus, the proof is concluded. □

In what follows, we extend Theorem 1 to the robust control problem.

B. Robust control via SOFC

Consider the following uncertain T-S descriptor model:

$$\begin{aligned} (E_v + \Delta E_v)x_{\kappa+} &= (A_h + \Delta A_h)x_\kappa + (B_h + \Delta B_h)u_\kappa \\ y_\kappa &= (C_h + \Delta C_h)x_\kappa, \end{aligned} \tag{13}$$

where $\Delta A_h = D_h^a \Delta_a(t) F_h^a$, $\Delta B_h = D_h^b \Delta_b(t) F_h^b$, $\Delta C_h = D_h^c \Delta_c(t) F_h^c$, and $\Delta E_v = D_v^e \Delta_e(t) F_v^e$ with $\Delta_a^T(t)\Delta_a(t) \le I$, $\Delta_b^T(t)\Delta_b(t) \le I$, $\Delta_c^T(t)\Delta_c(t) \le I$, and $\Delta_e^T(t)\Delta_e(t) \le I$.

The T-S descriptor system (13) and the control law (5) can be written as

$$\begin{bmatrix} \bar{A}_h & -\bar{E}_v & \bar{B}_h \\ \left(H_{hhh^-v}\right)^{-1} K_{hh^-v}\bar{C}_z & 0_{m\times n} & -I_m \end{bmatrix}\begin{bmatrix} x_\kappa \\ x_{\kappa+} \\ u_\kappa \end{bmatrix} = 0, \quad (14)$$

where $\bar{A}_h = A_h + \Delta A_h$, $\bar{B}_h = B_h + \Delta B_h$, $\bar{C}_h = C_h + \Delta C_h$, and $\bar{E}_v = E_v + \Delta E_v$.

Consider the delayed Lyapunov function (7). Through Finsler's Lemma, the variation (9) under constraint (14) renders

$$\Upsilon_{hhh^-v} + \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\Delta\bar{A} + \Delta\bar{A}^T \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}^T < 0, \quad (15)$$

with Υ_{hhh^-v} as (11) (see Theorem 1) and $\Delta\bar{A} = \begin{bmatrix} \Delta A_h & -\Delta E_v & \Delta B_h \\ \left(H_{hhh^-}\right)^{-1} K_{hh^-}\Delta C_h & 0_{m\times n} & 0_m \end{bmatrix}$. Matrices $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are defined in (12). Moreover, $\Delta\bar{A}$ can be separated as $\Delta\bar{A} = \bar{D}\bar{\Delta}\bar{F}$ with

$$\bar{D} = \begin{bmatrix} D_h^a & D_h^b & 0 & -D_v^e \\ 0 & 0 & \left(H_{hhh^-v}\right)^{-1} K_{hh^-v} D_h^c & 0 \end{bmatrix},$$

$$\bar{\Delta} = \begin{bmatrix} \Delta_a & 0 & 0 & 0 \\ 0 & \Delta_b & 0 & 0 \\ 0 & 0 & \Delta_c & 0 \\ 0 & 0 & 0 & \Delta_e \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} F_h^a & 0 & 0 \\ 0 & 0 & F_h^b \\ F_h^c & 0 & 0 \\ 0 & F_v^e & 0 \end{bmatrix}.$$

Then, expression (15) yields

$$\Upsilon_{hhh^-v} + \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D}\bar{\Delta}\bar{F} + \bar{F}^T\bar{\Delta}^T\bar{D}^T \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}^T < 0 . \quad (16)$$

Thanks to the bounds on the uncertainties, we have $\bar{\Delta}^T\bar{\Delta} \le I$.

Applying Property 1 with $M^T = \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D}\bar{\Delta}$, $R = \bar{F}$, and $Q = \bar{\mathrm{T}} = diag\left(\tau^a_{hh^-v}I, \tau^b_{hh^-v}I, \tau^c_{hh^-v}I, \tau^e_{hhh^-}I\right)$, $\bar{\mathrm{T}} = \bar{\mathrm{T}}^T > 0$, (16) writes

$$\Upsilon_{hhh^-v} + \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D}\bar{\Delta}\bar{\mathrm{T}}^{-1}\bar{\Delta}^T\bar{D}^T \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}^T + \bar{F}^T\bar{\mathrm{T}}\bar{F} < 0 . \quad (17)$$

Therefore, the following result can be stated:

Theorem 2: The uncertain T-S descriptor system (13) together with the control law (5) is asymptotically stable if there exist matrices $P_h = (P_h)^T > 0$, G_{hh^-}, H_{hhh^-v}, K_{hh^-v}, and scalars $\tau^a_{hh^-v} > 0$, $\tau^b_{hh^-v} > 0$, $\tau^c_{hh^-v} > 0$, $\tau^e_{hhh^-} > 0$, such that the following inequality holds

$$\Psi_{hhh^-v} = \left[\begin{array}{c:c:c} \Upsilon_{hhh^-v} & \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D} & \bar{F}^T\bar{\mathrm{T}} \\ \hdashline (*) & -\bar{\mathrm{T}} & 0 \\ \hdashline (*) & (*) & -\bar{\mathrm{T}} \end{array}\right] < 0, \quad (18)$$

where $\bar{D}$, $\bar{F}$, and $\bar{T}$ are as defined above.

Proof: Recall (17) and by the use of the Schur complement we have

$$\Psi_{hhh^-v} = \left[\begin{array}{c:c} \Upsilon_{hhh^-v} + \bar{F}^T\bar{\mathrm{T}}\left(\bar{\mathrm{T}}\right)^{-1}\bar{\mathrm{T}}\bar{F} & \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D} \\ \hdashline (*) & -\bar{\mathrm{T}} \end{array}\right] < 0. \quad (19)$$

Employing again the Schur complement in the block (1,1) of (19) gives (18).

Moreover, (18) can be cast as LMI since

$$\bar{F}^T\bar{\mathrm{T}} = \begin{bmatrix} \left(F_h^a\right)^T\tau^a_{hh^-v} & 0 & \left(F_h^c\right)^T\tau^c_{hh^-v} & 0 \\ 0 & \left(F_h^b\right)^T\tau^b_{hh^-v} & 0 & 0 \\ 0 & 0 & 0 & \left(F_v^e\right)^T\tau^e_{hhh^-} \end{bmatrix}$$

and

$$\begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix}\bar{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ G_{hh^-}D_h^a & G_{hh^-}D_h^b & \eta K_{hh^-v}D_h^c & -G_{hh^-}D_v^e \\ 0 & 0 & K_{hh^-v}D_h^c & 0 \end{bmatrix}$$

with this the proof is concluded. □

Remark 1: In order to obtain LMI formulation for Theorems 1 and 2, a relaxation lemma over the convex-sums must be applied.

Remark 2: Conditions in Theorems 1 and 2 are given in LMI form once the arbitrary matrix η is selected. Several choices can be made, e.g., $\eta = B_h$, $\eta = 0_{n\times m}$, or $\eta = \begin{bmatrix} I_m \\ 0_{(n-m)\times m} \end{bmatrix}$. Different choices could lead on different solution sets [28].

Remark 3: If $E_v = I_n$, with $\Delta E_v = 0$, the classical T-S model is recovered, i.e.,

$$\begin{aligned} x_{\kappa+} &= \left(A_h + \Delta A_h\right) x_\kappa + \left(B_h + \Delta B_h\right) u_\kappa \\ y_\kappa &= \left(C_h + \Delta C_h\right) x_\kappa . \end{aligned} \tag{20}$$

Corollary 1: The uncertain classical T-S system (20) together with the control law (5) is asymptotically stable if there exist matrices $P_h = \left(P_h\right)^T > 0$, G_{hh^-}, H_{hhh^-}, K_{hh^-}, and scalars $\tau^a_{hh^-} > 0$, $\tau^b_{hh^-} > 0$, $\tau^c_{hh^-} > 0$ such that the following inequality holds

$$\Psi_{hhh^-} = \left[\begin{array}{c:c:c} \Upsilon_{hhh^-} & \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \\ \mathcal{H} \end{bmatrix} \bar{D} & \bar{F}^T \bar{\mathrm{T}} \\ \hdashline (*) & -\bar{\mathrm{T}} & 0 \\ \hdashline (*) & (*) & -\bar{\mathrm{T}} \end{array}\right] < 0, \tag{21}$$

where $\bar{F} = \begin{bmatrix} F^a_h & 0 & 0 \\ 0 & 0 & F^b_h \\ F^c_h & 0 & 0 \end{bmatrix}$, $\bar{\mathrm{T}} = diag\left(\tau^a_{hh^-} I, \tau^b_{hh^-} I, \tau^c_{hh^-} I\right)$, and $\bar{D} = \begin{bmatrix} D^a_h & D^b_h & 0 \\ 0 & 0 & \left(H_{hhh^-}\right)^{-1} K_{hh^-} D^c_h \end{bmatrix}$.

The proof is similar to Theorem 2, therefore it is not repeated. □

IV. Examples

Example 1 (continued). Recall the nonlinear descriptor in Example 1; its T-S descriptor representation in the compact set $\Omega_x \in \left\{x : x_1 \in \mathbb{R}, |x_2| \le 2\right\}$ gives:

$A_1 = \begin{bmatrix} -0.5 & 1.2 \\ -1.2 & 0.5 \end{bmatrix}$, $A_2 = \begin{bmatrix} -0.5 & 0.8 \\ -1.2 & 0.5 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ -0.7 \end{bmatrix}^T$, $C_2 = \begin{bmatrix} 0 \\ 1.3 \end{bmatrix}^T$, $B_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $i = 1,2$, $E_1 = \begin{bmatrix} 0.9 & 0.3 \\ -0.7 & 1.1 \end{bmatrix}$, and $E_2 = \begin{bmatrix} 0.9 & -0.1 \\ -0.1 & 1.1 \end{bmatrix}$. In the right-hand side, MFs are $h_1 = x_2^2/4$ and $h_2 = 1 - h_1$. In the left-hand side the MFs are $v_1 = \left(x_2 + 2\right)/4$ and $v_2 = 1 - v_1$. These sets of MFs hold the convex sum property in Ω_x.

Employing the conditions in Theorem 1 of [28], no solution was found; while using conditions in Theorem 1 of this work with $\eta = B_h$, a SOFC can be designed.

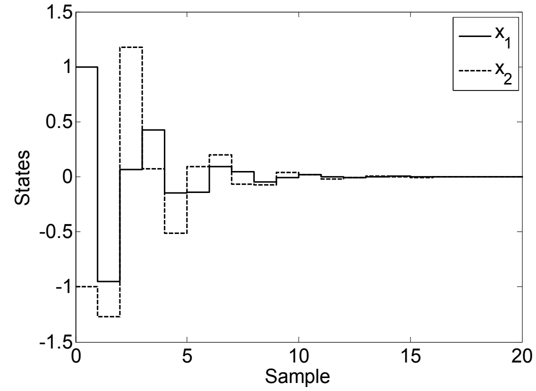

Figure 1. States evolution under the non-PDC control law in Example 1.

Due to the lack of space only some of the obtained values are displayed:

$P_1 = \begin{bmatrix} 0.48 & -0.08 \\ -0.08 & 0.25 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.60 & -0.14 \\ -0.14 & 0.28 \end{bmatrix}$, $K_{111} = -0.25$, $K_{222} = -0.10$, $K_{121} = -0.23$, $K_{122} = -0.27$, $H_{1111} = 0.27$, $H_{2222} = 0.30$, $H_{1122} = 0.28$, and $H_{1221} = 0.45$.

Simulation results are presented in Figure 1 for initial conditions $x(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ in Example 1.

Conditions in Theorem 1 with $\eta = B_h$ together with Lemma 1 are calculated with $r + r_e r^4 = 2 + (2)(2)^4 = 34$ LMI constraints. On the other hand for conditions in Theorem 1 in [28], the number of LMI conditions is $r + r^4 = 16 + (16)^4 = 65552$; this fact shows the importance of keeping the descriptor structure. ◊

The following numerical example compares the performance of Corollary 1 and Theorem 2 in [28], for the case when $E_v = I$. The example is adapted from [28], by including a real-valued parameter in the uncertain terms. For this example different values for the arbitrary matrix η are tested (see Remark 2).

Example 2. Consider a T-S model as in (20) with $r = 2$ and local matrices as follows [28]:

$$A_1 = \begin{bmatrix} 0.55 & 0.12 & 0.27 & 0.23 \\ 0.37 & 0.51 & -0.39 & 0.36 \\ -0.14 & -0.25 & 0.65 & 0.47 \\ -0.53 & -0.15 & 0.22 & 0.46 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.4 \\ -0.4 \\ 1.5 \\ 1.2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.62 & -0.29 & -0.31 & 0.28 \\ 0.24 & 0.59 & -0.23 & 0.19 \\ 0.19 & -0.37 & 0.43 & 0.15 \\ 0.16 & 0.31 & 0.22 & 0.55 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.25 \\ 0.20 \\ -0.35 \\ 0.20 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0 & 0.2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.41 & 0 & 0 & 0 \\ 0.50 & 0 & 0.7 & 0 \end{bmatrix},$$

$$D_i^a = \begin{bmatrix} 0.1 + a & 0.2 & 0 & 0 \end{bmatrix}^T, \quad F_i^a = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 \end{bmatrix},$$

$D_i^b = [0.1 \quad 0.1+a \quad 0 \quad 0.12]^T$, $F_i^b = 0.3$, $D_i^c = [0.1 \quad 0.1]^T$, $F_i^c = [0.1 \quad 0.1-a \quad 0 \quad 0.01]$, $i = 1,2$, and $a > 0$ parameter. The goal is to design a SOFC for as large values of a as possible. Table 1 summarizes the obtained results.

In Table 1, it can be seen that the parameter a is larger when Corollary 1 is applied, i.e., the new approach allows obtaining a larger size of the uncertainty than the one in [28]. ◊

Notice that in both examples, the matrix $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ is not full rank, thus the conditions in [25] cannot be applied. Moreover, in both examples the output is nonlinear.

TABLE I. RESULTS FOR EXAMPLE 2.

Approach	Maximum parameter value
Theorem 2 in [28] with $\eta = B_h$	$a = 0.1$
Theorem 2 in [28] with $\eta = 0_{n\times m}$	$a = 0.3$
Corollary 1 with $\eta = B_h$	$a = 1.0$
Corollary 1 with $\eta = 0_{n\times m}$	$a = 1.5$

V. CONCLUSION

The paper presents a new approach for SOFC design for nonlinear descriptor models via their T-S representation and extends the approach to robust control. The conditions are given in LMI form; moreover, it is possible to add ‘easily’ constraints on the input, decay rate, etc. Our further research is focus on the study of singular systems.

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