Observer design for Takagi-Sugeno descriptor models: an LMI approach ⋆

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Abstract

This paper considers observer design for nonlinear descriptor systems. We propose approaches based on Takagi-Sugeno (TS) models. An extended estimation vector is used in order to keep the descriptor structure of the observer. The design conditions for this new type of observers are expressed as LMI constraints. The proposed observer structure, via an intermediate variable as estimated variable, is able to recover the previous observer results for TS descriptors. Moreover, through a direct extension via the so-called Finsler's Lemma, relaxed conditions are obtained. Numerical examples show the effectiveness of the proposed approaches.

Key words: Linear matrix inequalities; Takagi-Sugeno models; nonlinear observer design; Finsler's Lemma.

1 Introduction

The natural behaviour of physical systems is described by nonlinear models. However, finding global conditions for the stability and stabilization of nonlinear systems is often laborious and difficult to perform. That is why the use of linear approximations is very common; however, this procedure provides local conclusions [1]. In recent years, the so-called Takagi-Sugeno (TS) models have been widely used to represent a large class of nonlinear models [2]. TS models are a collection of linear systems blended together by membership functions (MFs), which are nonlinear and share the convex-sum property [3]. An advantage of using TS models is that they are able to exactly represent a nonlinear model in a compact set of the state space, via the sector nonlinearity approach [4]. Moreover, the stability analysis and controller design can be performed in terms of linear matrix inequalities (LMIs), which can be efficiently solved through convex optimization techniques [5,6]. Nevertheless, employing the sector nonlinearity approach, the number of linear models is exponentially related to the number of nonlinearities in the nonlinear model [4].

A wide class of physical systems can be written as descriptor models [7]. Since this type of system often appears in control problems, in [8] a TS descriptor model was introduced; a TS descriptor model allows obtaining a smaller number of LMI constraints [9–12] because it introduces a sector nonlinearity approach for the left-hand side and preserves the structure of the nonlinear model. From a computational point of view, a regular descriptor system allows using classical ODE solvers [7,13].

Generally, the state vector is partially unknown, thus an observer or estimator can be implemented [14,15]. During the last years nonlinear observers have been addressed via TS models [16–20]. Two different cases can be considered: 1) the premise vector depends only on the measured variables; 2) the premise vector depends also on states that must be estimated [4,21]. This work considers the first case.

For observer design for descriptor models very few results exist. For linear models [22] and [23] present an approach for functional observers design based on the gen-

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eralized Sylvester equation or defining partial impulse observability. To guarantee global convergence of the estimation error, observers for TS descriptor models have been proposed in [10,13]. The conditions are obtained via a quadratic Lyapunov function. The main drawback is that the conditions are expressed in terms of bilinear matrix inequalities (BMIs).

This paper provides a way to obtain LMI conditions for the observer design of TS regular descriptor models. The formulation of the LMI problem is linked to a new observer form. The results are enhanced via the so-called Finsler's Lemma [24,25].

The paper is organized as follows: Section 2 introduces the TS descriptor model, some lemmas, and previous work; Section 3 states the main result with a full LMI approach, ways to relax these LMI conditions, and it illustrates the improvement brought by the new approaches through an example; Section 4 concludes the paper.

Notation: The following shorthand notation is adopted to represent convex sums of matrix expressions: $\Upsilon_h = \sum_{i=1}^{r} h_i(z(t))\Upsilon_i$, $\Upsilon_h^{-1} = \left(\sum_{i=1}^{r} h_i(z(t))\Upsilon_i\right)^{-1}$, $\Upsilon_{hh}^{v} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} h_i(z(t))\,h_j(z(t))\,v_k(z(t))\,\Upsilon_{ij}^{k}$. Subscripts may change to v if the respective MF is $v_k(z(t))$. An asterisk $(*)$ will be used in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side. Arguments will be omitted when their meaning is straightforward.

2 Problem statement

Consider the following descriptor nonlinear system:

$$\begin{aligned} E(x(t))\dot{x}(t) &= A(x(t))x(t) + B(x(t))u(t) \\ y(t) &= C(x(t))x(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ the control input, and $y(t) \in \mathbb{R}^o$ the output vector; $A(x)$, $B(x)$, $C(x)$ are matrices of appropriate sizes while $E(x)$ is a nonsingular matrix for all $x(t)$ in the considered compact set of the state space Ω. In mechanical systems, the matrix $E(x)$ that contains the inertia matrix is invertible and in most cases it appears as a diagonal matrix [13,26]. Moreover, a nonsingular matrix $E(x)$ allows using classical ODE solvers. Applying the sector nonlinearity approach [4], the p nonlinearities in the right-hand side of (1) and the p_e nonlinearities in the left-hand side of (1) are captured via membership functions (MFs). These MFs have the convex-sum property in the compact set Ω, i.e., $\sum_{i=1}^{r} h_i(z) = 1$, $h_i(z) \geq 0$, $\sum_{k=1}^{r_e} v_k(z) = 1$, $v_k(z) \geq 0$; with $r = 2^p$, $r_e = 2^{p_e}$, and $z(t)$ is the premise vector depending on measured variables.

The nonlinear descriptor system (1) can be rewritten as:

$$\begin{aligned} \sum_{k=1}^{r_e} v_k(z(t))\,E_k\dot{x}(t) &= \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) \\ y(t) &= \sum_{i=1}^{r} h_i(z(t))\,C_i x(t), \end{aligned} \tag{2}$$

where matrices A_i, B_i, C_i, and E_k represent the i-th linear right-hand side model (2) and the k-th linear left-hand side model of the TS descriptor model.

Keeping the descriptor structure can significantly reduce the number of models as well as the number of LMIs; thus, it may increase the feasibility set; therefore it lowers the conservativeness [4]. The following example points out these remarks.

Example 1 *Consider the nonlinear descriptor system (1) with* $u(t) = 0$ *and matrices* $E(x) = \begin{bmatrix} 1 & \dfrac{-1}{1+x_1^2} \\ \dfrac{1}{1+x_1^2} & 1 \end{bmatrix}$, $A(x) = \begin{bmatrix} sin(x_1/x_1 & -2 \\ 4 & -cos(x_1) \end{bmatrix}$, *and the inverse of matrix* $E(x)$ *gives* $E^{-1}(x) = \eta\begin{bmatrix} (1+x_1^2) & 1 \\ -1 & (1+x_1^2) \end{bmatrix}$, $\eta = (1+x_1^2)/(2+2x_1^2+x_1^4)$. *The representation in the form (2) gives* $r_e = 2^1 = 2$ *due to the nonlinear term* $1/(1+x_1^2)$ *in* $E(x)$ *and* $r = 2^2 = 4$ *due to* $cos(x_1)$ *and* $sin(x_1)/x_1$*; note that the TS descriptor has global sector nonlinearity, i.e., it means that (2) is equivalent to (1) in* $\mathbb{R}^2$. *To rewrite (1) into the classical TS representation it is necessary to invert the matrix* $E(x)$, *resulting in* $\dot{x}(t) = E^{-1}(x)A(x)x$. *This results in* $r = 2^4 = 16$ *since all the nonlinearities are on the right-hand side. Under the quadratic framework, the number of LMI conditions to be verified for the classical TS model is* $16^2 + 1 = 257$ *while for the TS descriptor one is* $(2) \times (4)^2 + 1 = 33$.

One way to deal with matrix E_v on the left-hand side is to treat the TS descriptor model (2) as follows:

$$0\ddot{x} = A_h x + B_h u - E_v \dot{x}.$$

Therefore, the following equivalent representation of (2) with $\bar{x} = \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix}^T$ can be stated [9]:

$$\bar{E}\dot{\bar{x}} = \bar{A}_{hv}\bar{x} + \bar{B}_h u, \quad y = \bar{C}_h \bar{x}, \tag{3}$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, $\bar{B}_h = \begin{bmatrix} 0 \\ B_h \end{bmatrix}$, and $\bar{C}_h = \begin{bmatrix} C_h & 0 \end{bmatrix}$. Note that this is a rewriting of the regular

TS descriptor model (2).

Convex sums will appear whose MFs have to be dropped out in order to obtain LMI conditions. To this end, the following result has been chosen because it does not involve slack variables and it provides a good compromise between numerical complexity and quality of solutions.

Lemma 1 *(Relaxation Lemma)[27]. Let Υ_{ij}^k be matrices of appropriate dimensions where $i, j \in \{1, \ldots, r\}$, $k \in \{1, \ldots, r_e\}$. Then $\Upsilon_{hh}^v < 0$, holds if*

$$\Upsilon_{ii}^k < 0, \ \forall i, \quad \frac{2}{r-1}\Upsilon_{ii}^k + \Upsilon_{ij}^k + \Upsilon_{ji}^k < 0, \ i \neq j. \tag{4}$$

Relaxations with different degrees of conservatism and/or complexity [28–30] apply directly.

Lemma 2 *(Finsler's Lemma) [24]: Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(R) < n$; the following expression are equivalent:*
a) $x^T Q x < 0, \forall x \in \{x \in \mathbb{R}^n : x \neq 0, Rx = 0\}$.
b) $\exists X \in \mathbb{R}^{n\times m} : Q + XR + R^T X^T < 0$.

2.1 Previous work

For linear descriptor systems the observation problem was addressed in [22,23,31]. The solution for the continuous-time case is expressed in terms of a set linear equations and rank tests. In [31] the discrete-time case is presented, the observer design needs to solve a linear equation and one LMI. In all these results, the final observer is not in descriptor form. In [10,13] the following observer for the system (3) was proposed:

$$\bar{E}\dot{\hat{\bar{x}}}^* = \bar{A}_{hv}\hat{\bar{x}}^* + \bar{B}_h u + \bar{L}_{hv}(y - \hat{y}), \quad \hat{y} = \bar{C}_h \hat{\bar{x}}^*, \tag{5}$$

with $\hat{\bar{x}}^* = \begin{bmatrix}\hat{x}^T & \dot{\hat{x}}^T\end{bmatrix}^T$ and $\bar{L}_{hv} = \begin{bmatrix}0 & L_{hv}^T\end{bmatrix}^T$. The main task is to make the estimation error $e = x - \hat{x}$ converge to zero when $t \to \infty$. For this purpose an extended estimation error is defined: $e^* = \bar{x} - \hat{\bar{x}}^* = \begin{bmatrix} x - \hat{x} \\ \dot{x} - \dot{\hat{x}}\end{bmatrix}$ and its dynamic is given as $\bar{E}\dot{e}^* = \left(\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h\right)e^*$. This representation is commonly used in the TS descriptor literature [9,11].

In [10], the following Lyapunov function is used:

$$V(e^*) = e^{*T}\bar{E}^T P e^*, \quad \bar{E}^T P = P^T \bar{E} \geq 0, \tag{6}$$

with $P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4\end{bmatrix}$, $P_1 = P_1^T > 0$; this special structure on P allows adding slack variables; moreover, developing it results in $V(e^*) = e^T P_1 e > 0$. In [10,13], they obtain the following inequality:

$$\begin{gathered} P^T\bar{A}_{hv} - P^T\bar{L}_{hv}\bar{C}_h + (*) < 0 \quad \Leftrightarrow \\ \begin{bmatrix} P_3^T A_h - P_3^T L_{hv} C_h + (*) & (*) \\ P_4^T A_h - P_4^T L_{hv} C_h + P_1 - E_v^T P_3 & -P_4^T E_v + (*) \end{bmatrix} < 0 \end{gathered} \tag{7}$$

which is not LMI because of the terms $P_3^T L_{hv} C_h$ and $P_4^T L_{hv} C_h$. To achieve LMI conditions one way is to fix P_3, for example $P_3 = 0$ or $P_3 = P_4$ [10]; another possibility is use a two-step algorithm: design the gains L_{jk} and use (7) to verify the convergence of the estimation error [13]. The final observer form is:

$$E_v\dot{\hat{x}} = A_h\hat{x} + B_h u + L_{hv}(y - \hat{y}), \quad \hat{y} = C_h\hat{x}. \tag{8}$$

Our goal is to overcome the BMI problem in (7).

3 Main Result

In this section two approaches are presented for observer design via LMIs. The first one employs a full observer gain and a new structure for the final observer; the second approach provides extra degrees of freedom with respect to the first one by using Finsler's Lemma.

Initial Remark: An interesting approach would be to use a full observer gain, i.e., $\bar{L}_{hv} = \begin{bmatrix}L_{1hv}^T & L_{2hv}^T\end{bmatrix}^T$. This cannot be done directly with the structure of (5). Effectively, (5) writes

$$\begin{aligned}\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\hat{x}} \\ \ddot{\hat{x}}\end{bmatrix} = \begin{bmatrix} 0 & I \\ A_h & -E_v\end{bmatrix}\begin{bmatrix}\hat{x} \\ \dot{\hat{x}}\end{bmatrix} + \begin{bmatrix}0 \\ B_h\end{bmatrix}u \\ + \begin{bmatrix}L_{1hv} \\ L_{2hv}\end{bmatrix}\begin{bmatrix}C_h & 0\end{bmatrix}\begin{bmatrix}x - \hat{x} \\ \dot{x} - \dot{\hat{x}}\end{bmatrix}.\end{aligned} \tag{9}$$

In (9), the state vector is $\hat{\bar{x}}^* = \begin{bmatrix}\hat{x}^T & \dot{\hat{x}}^T\end{bmatrix}^T$, thus $\dot{\hat{\bar{x}}}^* = \begin{bmatrix}\dot{\hat{x}}^T & \ddot{\hat{x}}^T\end{bmatrix}^T$ and the first row of (9) implies

$$\dot{\hat{x}} = \dot{\hat{x}} + L_{1hv}C_h(x - \hat{x}), \tag{10}$$

which is consistent only if $x - \hat{x} = 0$ or if $L_{1hv}C_h = 0$. Of course, with $L_{1hv}C_h = 0$ the observer (5) is recovered. This shows that to use a full observer gain, the estimated state vector must be changed. Therefore, in order to ensure consistency on the observer equations, a new estimated state vector will be introduced: $\hat{\bar{x}} = \begin{bmatrix}\hat{x}^T & \beta^T\end{bmatrix}^T$; where β will be defined later on (see (23)). The main idea is that $\beta \to \dot{x}$ as $t \to \infty$.

3.1 Approach 1: LMI conditions for observer design

Based on the remark above the following observer is proposed:

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + P_h^{-T}\bar{L}_{hv}\left(y-\hat{y}\right), \quad \hat{y} = \bar{C}_h\hat{\bar{x}}, \tag{11}$$

where $\hat{\bar{x}} = \begin{bmatrix}\hat{x}^T & \beta^T\end{bmatrix}^T$, $\bar{L}_{hv} = \begin{bmatrix}L_{1hv}^T & L_{2hv}^T\end{bmatrix}^T$ and the structure of $P_h = \begin{bmatrix}P_1 & 0\\ P_{3h} & P_{4h}\end{bmatrix}$, $P_1 = P_1^T > 0$, P_{4h} being a nonsingular matrix; therefore $P_h^{-1} = \begin{bmatrix}P_1^{-1} & 0\\ -P_{4h}^{-1}P_{3h}P_1^{-1} & P_{4h}^{-1}\end{bmatrix}$. This form will bring a new expression for the final observer (16) and is the main way to obtain LMI constraints.

An extended estimation error is defined as:

$$\bar{e} = \bar{x} - \hat{\bar{x}} = \begin{bmatrix}x-\hat{x}\\ \dot{x}-\beta\end{bmatrix}. \tag{12}$$

Its dynamic is

$$\bar{E}\dot{\bar{e}} = \left(\bar{A}_{hv} - P_h^{-T}\bar{L}_{hv}\bar{C}_h\right)\bar{e}. \tag{13}$$

Consider the following Lyapunov function candidate:

$$V\left(\bar{e}\right) = \bar{e}^T\bar{E}^T P_h\bar{e}, \quad \bar{E}^T P_h = P_h^T\bar{E} \geq 0. \tag{14}$$

Then, the following result can be stated.

Theorem 1 *Consider the model (3) together with the observer (11). If there exist matrices $P_1 = P_1^T > 0$, P_{3j}, P_{4j}, L_{1jk}, and L_{2jk}, $j \in \{1,\dots,r\}$, $k \in \{1,\dots,r_e\}$ such that (4) holds with*

$$\Upsilon_{ij}^k = \begin{bmatrix}P_{3j}^T A_i - L_{1jk}C_i + (*) & (*)\\ P_{4j}^T A_i - L_{2jk}C_i + P_1 - E_k^T P_{3j} & -P_{4j}^T E_k + (*)\end{bmatrix} \tag{15}$$

then the estimation error e is asymptotically stable. Moreover, the final observer structure is

$$\begin{aligned} E_v\dot{\hat{x}} &= A_h\hat{x} + B_h u + \begin{bmatrix}E_v & I\end{bmatrix}P_h^{-T}\begin{bmatrix}L_{1hv}\\ L_{2hv}\end{bmatrix}\left(y-\hat{y}\right)\\ \hat{y} &= C_h\hat{x}. \end{aligned} \tag{16}$$

Proof. The time-derivative of the Lyapunov function (14) is:

$$\dot{V}(\bar{e}) = \dot{\bar{e}}^T\bar{E}^T P_h\bar{e} + \bar{e}^T P_h^T\bar{E}\dot{\bar{e}} + \bar{e}^T\bar{E}^T\dot{P}_h\bar{e} < 0. \tag{17}$$

Considering that $\bar{E}^T\dot{P}_h = \begin{bmatrix}\dot{P}_1 & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, we have

$$\begin{aligned} &\dot{V}(\bar{e}) < 0\\ &\Leftrightarrow \bar{e}^T\left(\bar{A}_{hv} - P_h^{-T}\bar{L}_{hv}\bar{C}_h\right)^T P_h\bar{e} + (*) < 0\\ &\Leftrightarrow P_h^T\bar{A}_{hv} - \bar{L}_{hv}\bar{C}_h + (*) < 0\\ &\Leftrightarrow \begin{bmatrix}P_{3h}^T A_h - L_{1hv}C_h + (*) & (*)\\ \begin{pmatrix}P_{4h}^T A_h - L_{2hv}C_h\\ +P_1 - E_v^T P_{3h}\end{pmatrix} & -P_{4h}^T E_v + (*)\end{bmatrix} < 0, \end{aligned} \tag{18}$$

which corresponds to (15) via Lemma 1. The regularity of P_{4h} in the compact set Ω is given as follows: if inequality (18) holds, it ensures the condition $-P_{4h}^T E_v - E_v^T P_{4h} < 0$. Now, recall that E_v is nonsingular ($E_v x \neq 0$, $\forall x \neq 0$) and let us assume that P_{4h} is singular, therefore it exists $x \neq 0$ such as $P_{4h}x = 0$; for such a $x \neq 0$ we have $x^T\left(-P_{4h}^T E_v - E_v^T P_{4h}\right)x = 0$ which contradicts the condition $-P_{4h}^T E_v - E_v^T P_{4h} < 0$. Thus if $\Upsilon_{hh}^v < 0$ is true then P_{4h} is nonsingular.

To obtain the final form (16), recall (11), i.e.,

$$\begin{aligned} \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\hat{x}}\\ \dot{\beta}\end{bmatrix} &= \begin{bmatrix}0 & I\\ A_h & -E_v\end{bmatrix}\begin{bmatrix}\hat{x}\\ \beta\end{bmatrix} + \begin{bmatrix}0\\ B_h\end{bmatrix}u\\ &\quad + P_h^{-T}\begin{bmatrix}L_{1hv}\\ L_{2hv}\end{bmatrix}\left(y-\hat{y}\right). \end{aligned} \tag{19}$$

Define

$$\begin{aligned} \begin{bmatrix}\mathcal{K}_{1hhv}(h)\\ \mathcal{K}_{2hv}(h)\end{bmatrix} &= \begin{bmatrix}P_1^{-1} & -P_1^{-1}P_{3h}^T P_{4h}^{-T}\\ 0 & P_{4h}^{-T}\end{bmatrix}\begin{bmatrix}L_{1hv}\\ L_{2hv}\end{bmatrix}\\ &= \begin{bmatrix}P_1^{-1}\left(L_{1hv} - P_{3h}^T P_{4h}^{-T}L_{2hv}\right)\\ P_{4h}^{-T}L_{2hv}\end{bmatrix}, \end{aligned} \tag{20}$$

where the subscripts h v stand for dependence on convex structures, while $'(h)'$ means dependence on non-convex structures, for instance: $\mathcal{K}_{2hv}(h) = \sum_{j=1}^{r}\sum_{k=1}^{r_e} h_j\left(z\right)v_k\left(z\right)\left(\sum_{j=1}^{r} h_j\left(z\right)P_{4j}\right)^{-T} L_{2jk}$.

Thus (19) writes:

$$\begin{aligned} \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\hat{x}}\\ \dot{\beta}\end{bmatrix} &= \begin{bmatrix}0 & I\\ A_h & -E_v\end{bmatrix}\begin{bmatrix}\hat{x}\\ \beta\end{bmatrix} + \begin{bmatrix}0\\ B_h\end{bmatrix}u\\ &\quad + \begin{bmatrix}\mathcal{K}_{1hhv}(h)C_h\\ \mathcal{K}_{2hv}(h)C_h\end{bmatrix}\left(x-\hat{x}\right), \end{aligned} \tag{21}$$

or equivalently

$$\begin{aligned}\dot{\hat{x}} &= \beta + \mathcal{K}_{1hhv}(h)C_h\left(x-\hat{x}\right) && \text{(22a)}\\ E_v\beta &= A_h\hat{x} + B_h u + \mathcal{K}_{2hv}(h)C_h\left(x-\hat{x}\right). && \text{(22b)}\end{aligned}$$

From (22a):

$$\beta = \dot{\hat{x}} - \mathcal{K}_{1hhv}(h)C_h\left(x-\hat{x}\right). \quad (23)$$

Remark 1 *Equation (23) provides the definition of the intermediate variable β that plays a role similar to $\dot{\hat{x}}$ in (5). If the observation error $\bar{e}$ defined in (13) converges, i.e., $\bar{e} \to 0$, then $\hat{x} \to x$ and according to (23) $\beta \to \dot{\hat{x}}$.*

Finally, eliminating the intermediate variable β gives the following observer structure:

$$\begin{aligned}E_v\dot{\hat{x}} = {} & A_h\hat{x} + B_h u + \mathcal{K}_{2hv}(h)C_h\left(x-\hat{x}\right)\\ & + E_v\mathcal{K}_{1hhv}(h)C_h\left(x-\hat{x}\right). \quad (24)\end{aligned}$$

Regrouping the terms in (24) and using the definition (20), the final observer form (16) is obtained, thus concluding the proof. □

3.2 Approach 2: improvements via Finsler's Lemma

In this approach, Finsler's Lemma is used to decouple the Lyapunov function from the observer expression. This allows adding slack variables. The proposed observer in descriptor form is

$$\bar{E}\dot{\hat{\bar{x}}} = \bar{A}_{hv}\hat{\bar{x}} + \bar{B}_h u + Y_{hv}^{-T}\bar{L}_{hvv}\left(y-\hat{y}\right), \quad \hat{y} = \bar{C}_h\hat{\bar{x}}, \quad (25)$$

with $Y_{hv} = \begin{bmatrix} P_1 & 0 \\ Y_{3hv} & Y_{4hv}\end{bmatrix}$ and $\bar{L}_{hvv} = \begin{bmatrix} L_{1hvv} \\ L_{2hvv}\end{bmatrix}$.

For this case, the dynamic of the estimation error (12) is:

$$\begin{aligned}\bar{E}\dot{\bar{e}} &= \left(\bar{A}_{hv} - Y_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h\right)\bar{e}\\ &\Leftrightarrow \begin{bmatrix}\bar{A}_{hv} - Y_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h & -I\end{bmatrix}\begin{bmatrix}\bar{e}\\ \bar{E}\dot{\bar{e}}\end{bmatrix} = 0 \quad (26)\end{aligned}$$

Consider the non-quadratic Lyapunov function with a more general form: $V\left(\bar{e}\right) = \bar{e}^T\bar{E}^T P_{hhv}\bar{e}$, $P_{hhv} = \begin{bmatrix} P_1 & 0 \\ P_{3hhv} & P_{4hhv}\end{bmatrix}$. Then, the following result can be stated.

Theorem 2 *Consider the system (3) together with the observer (25). If there exist matrices $P_1 = P_1^T > 0$, P_{3ijl}, P_{4ijl}, Y_{3jl}, Y_{4jl}, L_{1jkl}, and L_{2jkl}, $i,j \in \{1,\dots,r\}$ and $k,l \in \{1,\dots,r_e\}$ such that (27)∽(30) hold*

$$\Upsilon_{ii}^{kk} < 0, \quad \forall i,k \quad (27)$$

$$\frac{2}{r_e-1}\Upsilon_{ii}^{kk} + \Upsilon_{ii}^{kl} + \Upsilon_{ii}^{lk} < 0, \quad \forall i, \quad k \neq l \quad (28)$$

$$\frac{2}{r-1}\Upsilon_{ii}^{kk} + \Upsilon_{ij}^{kk} + \Upsilon_{ji}^{kk} < 0, \quad \forall k, \quad i \neq j \quad (29)$$

$$\begin{aligned}&\frac{4}{(r-1)(r_e-1)}\Upsilon_{ii}^{kk} + \frac{2}{r_e-1}\left(\Upsilon_{ij}^{kk} + \Upsilon_{ji}^{kk}\right)\\ &\quad + \frac{2}{r-1}\left(\Upsilon_{ii}^{kl} + \Upsilon_{ii}^{lk}\right) + \Upsilon_{ij}^{kl} + \Upsilon_{ji}^{kl}\\ &\quad + \Upsilon_{ij}^{lk} + \Upsilon_{ji}^{lk} < 0, \quad k \neq l, \quad i \neq j \quad (30)\end{aligned}$$

with Υ_{ij}^{kl} defined in (32). Then the estimation error e is asymptotically stable. The observer structure is:

$$\begin{aligned}E_v\dot{\hat{x}} &= A_h\hat{x} + B_h u + \begin{bmatrix}E_v & I\end{bmatrix} Y_{hv}^{-T}\begin{bmatrix}L_{1hvv}\\ L_{2hvv}\end{bmatrix}\left(y-\hat{y}\right)\\ \hat{y} &= C_h\hat{x}. \quad (31)\end{aligned}$$

Proof. The time-derivative of the Lyapunov function is

$$\dot{V}\left(\bar{e}\right) = \begin{bmatrix}\bar{e}\\ \bar{E}\dot{\bar{e}}\end{bmatrix}^T\begin{bmatrix}0 & P_{hhv}^T\\ P_{hhv} & 0\end{bmatrix}\begin{bmatrix}\bar{e}\\ \bar{E}\dot{\bar{e}}\end{bmatrix} < 0. \quad (33)$$

Using Lemma 2, inequality (33) under constraint (26) yields

$$\begin{bmatrix}0 & P_{hhv}^T\\ P_{hhv} & 0\end{bmatrix} + \begin{bmatrix}\mathcal{U}_{(.)}\\ \mathcal{V}_{(.)}\end{bmatrix}\begin{bmatrix}\bar{A}_{hv} - Y_{hv}^{-T}\bar{L}_{hvv}\bar{C}_h & -I\end{bmatrix} + (*) < 0. \quad (34)$$

Choosing $\mathcal{U}_{(.)} = Y_{hv}^T$, $\mathcal{V}_{(.)} = \epsilon G_{(.)}^{-T}Y_{hv}^T$ and applying the congruence property with $diag\begin{bmatrix}I, G_{(.)}^T\end{bmatrix}$, (34) gives:

$$\begin{bmatrix} Y_{hv}^T\bar{A}_{hv} - \bar{L}_{hvv}\bar{C}_h + (*) & (*)\\ \begin{pmatrix}\epsilon\left(Y_{hv}^T\bar{A}_{hv} - \bar{L}_{hvv}\bar{C}_h\right)\\ +G_{(.)}^T\left(P_{hhv} - Y_{hv}\right)\end{pmatrix} & -\epsilon Y_{hv}^T G_{(.)} + (*)\end{bmatrix} < 0. \quad (35)$$

Let matrix $G_{(.)}$ be defined as $G_{hvv} = \begin{bmatrix} I & -P_1^{-1}Y_{3hv}^T E_v\\ 0 & E_v\end{bmatrix}$.

Via an extension of Lemma 1, inequality (35) gives (27)∽(30). The final form of the observer (31) can be obtained via similar manipulations as in Theorem 1. The proof of the regularity of matrices P_{4hhv} and Y_{4hv} follows the same lines as Theorem 1.□

Remark 2 *The conditions in Theorem 2 are LMIs for a given scalar ϵ [32,33]. In order to avoid any opti-*

$$
\Upsilon_{ij}^{kl} = \begin{bmatrix} Y_{3jl}^T A_i - L_{1jkl} C_i + (*) & (*) & (*) & (*) \\ Y_{4jl}^T A_i - L_{2jkl} C_i + P_1 - E_k^T Y_{3jl} & -Y_{4jl}^T E_k + (*) & (*) & (*) \\ \epsilon \left(Y_{3jl}^T A_i - L_{1jkl} C_i \right) & \epsilon \left(P_1 - Y_{3jl}^T E_k \right) & -2\epsilon P_1 & 0 \\ \epsilon \left(Y_{4jl}^T A_i - L_{2jkl} C_i \right) + E_k^T \left(P_{3ijl} - Y_{3jl} \right) & -\epsilon Y_{4jl}^T E_k + E_k^T \left(P_{4ijl} - Y_{4jl} \right) & 0 & -\epsilon Y_{4jl}^T E_k + (*) \end{bmatrix}, \tag{32}
$$

mization technique to search for such an ϵ, a logarithmically spaced search has been proposed in [34]. They use a finite set of LMI constraint problems with $\epsilon \in \{10^{-6}, 10^{-5}, \dots, 1, \dots, 10^6\}$. Thus, ϵ is fixed ant it is not a decision variable. This logarithmically spaced search has been tested intensively in [34–36] .

Remark 3 *Conditions in Theorem 2 involve four convex-sums, two on MFs $h(\bullet)$ and two on MFs $v(\bullet)$; this leads to a larger number of LMIs than Theorem 1. To obtain the same number of LMIs as Theorem 1, one should choose*

$$
P_{hh} = \begin{bmatrix} P_1 & 0 \\ P_{3hh} & P_{4hh} \end{bmatrix} \; Y_{hv} = \begin{bmatrix} P_1 & 0 \\ Y_{3h} & Y_{4h} \end{bmatrix} \; \bar{L}_{hv} = \begin{bmatrix} L_{1hv} \\ L_{2hv} \end{bmatrix}
$$

The fact that Finsler's Lemma allows adding slack variables can be used to also increment the number of convex sums. The configuration chosen for the matrices in Theorem 2 is done because it allows including convex structures from both sides of the TS descriptor model, i.e., the nonlinearities captured via MFs $h(\bullet)$ and $v(\bullet)$.

Corollary 1 *Results of Theorem 1 are always included in those of Theorem 2 under the same relaxation scheme.*

Proof. Suppose conditions of Theorem 1 hold: $\Upsilon_{hhv}^{Th1} = \begin{bmatrix} P_{3h}^T A_h - L_{1hv} C_h + (*) & (*) \\ P_{4h}^T A_h - L_{2hv} C_h + P_1 - E_v^T P_{3h} & -P_{4h}^T E_v + (*) \end{bmatrix}$.
Choose for Theorem 2: $P_{hhv} = Y_{hv} = P_h^{Th1}$ and $\bar{L}_{hvv} = \bar{L}_{hv}^{Th1}$, thus reducing it to:

$$
\begin{bmatrix} \Upsilon_{hhv}^{Th1} & (*) \\ \epsilon \Phi_{hhv} & -\epsilon \begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix} \end{bmatrix} < 0. \tag{36}
$$

with $\Phi_{hhv} = \begin{bmatrix} P_{3h}^T A_h - L_{1hv} C_h & P_1 - P_{3h}^T E_v \\ P_{4h}^T A_h - L_{2hv} C_h & -P_{4h}^T E_v \end{bmatrix}$. From Theorem 1, we have $\begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix} > 0$, thus via Schur's complement (36) is equivalent to:

$$
\Upsilon_{hhv}^{Th1} + \underbrace{\epsilon \, \Phi_{hhv}^T \begin{bmatrix} 2P_1 & 0 \\ 0 & P_{4h}^T E_v + E_v^T P_{4h} \end{bmatrix}^{-1} \Phi_{hhv}}_{\Gamma_{hhv}} < 0 \tag{37}
$$

If Theorem 1 holds, then it always exists a sufficiently small $\epsilon > 0$ such that (37) is true, (36) is true and Theorem 2 holds. □

Example 2 *Consider (2) with $r = r_e = 2$, $u = 0$, and the following matrices $E_1 = \begin{bmatrix} 1.1 & -0.1 \\ -0.2-b & 1.5 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0.9 & -0.1 \\ 0.2 & 0.2 \end{bmatrix}$, $A_1 = \begin{bmatrix} -0.2 & -1 \\ -0.1 & -1.9 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1+a & 0.6 \\ 1.7 & -0.3 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & -1 \end{bmatrix}$, and $C_2 = \begin{bmatrix} 0 & -0.6 \end{bmatrix}$; with parameters $a \in \left[-0.5, 2\right]$ and $b \in \left[-1, 1\right]$. The state x_2 is available while x_1 is unknown. MFs are defined as follows: $v_1 = 1/(1 + x_2^2)$, $v_2 = 1 - v_1$, $h_1 = x_2^2/4$, and $h_2 = 1 - h_1$. Figure 1 illustrates the feasible parameter values when using conditions in [10], Theorem 1, and Theorem 2.*

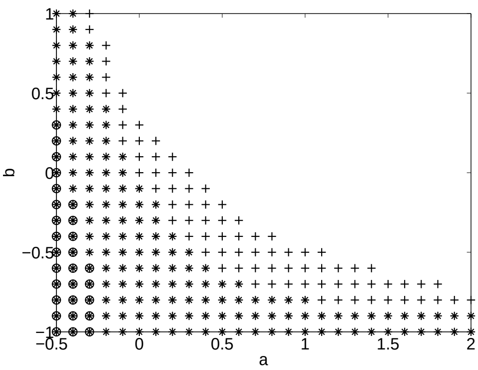

Fig. 1. Feasible sets for [10] (O), Theorem 1 ($\times$) and Theorem 2 ($+$) in Example 2.

For this example, choosing $(a, b) = (0.5, -0.2)$ there is no solution for the conditions of Theorem 1 while Theorem 2 provides a solution. For intance, for $\epsilon = 0.0001$ some *of the resulting matrices are:* $P_1 = \begin{bmatrix} 31.95 & -31.47 \\ -31.47 & 277.46 \end{bmatrix}$,

$$\begin{bmatrix} \bar{L}_{111}^T \\ \bar{L}_{112}^T \\ \bar{L}_{212}^T \\ \bar{L}_{222}^T \end{bmatrix} = \begin{bmatrix} -34.99 & 5.10 & -103.29 & -137.78 \\ -164.66 & 190.30 & -344.70 & 7.95 \\ -38.04 & 467.99 & -578.99 & -331.88 \\ -48.52 & 280.24 & -235.74 & -339.89 \end{bmatrix}.$$

Table 1 compares, in terms of the number of decision variables and the number of LMIs, the conditions in [13], Theorem 1, Theorem 2, and the option in Remark 3; where n is the number of states, o is the number of outputs, r is the number of rules on the the right-hand side, and r_e is the number of rules on the left-hand side. Note that the number of LMIs for Theorem 1 and Remark 3 is the same. The number of decision variables and LMIs in Theorem 2 is larger, thanks to Finsler's Lemma, which allows increasing the number of decision variables at the price of increasing the number of LMIs.

Table 1
Comparison between [13], Theorems 1 and 2, Remark 3.

Approach	No. of decision variables	No. of LMIs
[13]	$\frac{n(n+1)}{2} + n^2$ $+(n \times o) \times (r \times r_e)$	$r_e \times r^2 + 1$
Theorem 1	$\frac{n(n+1)}{2} \times r$ $+2(n \times o) \times (r \times r_e)$	$r_e \times r^2 + 1$
Theorem 2	$\frac{n(n+1)}{2} + 2n^2 \times (r^2 \times r_e)$ $+2(n \times o) \times (r \times r_e^2)$ $+2n^2 \times (r \times r_e)$	$r_e^2 \times r^2 + 1$
Remark 3	$\frac{n(n+1)}{2} + 2n^2 \times r^2$ $+2(n \times o) \times (r \times r_e)$ $+2n^2 \times r$	$r_e \times r^2 + 1$

4 Conclusions

A novel observer design of nonlinear descriptor systems represented by TS models has been presented. By introducing a new extended estimation vector it is possible to obtain new observer structures. Via this special observer structure an LMI formulation is available, thus overcoming existing results in the literature. A refinement has been proposed through Finsler's Lemma which gives BMI conditions; an algorithm is provided to overcome this issue. Moreover, our approaches consider nonlinear output matrices. The validity of the proposed approaches is illustrated via numerical examples.

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