Switching fuzzy observers for periodic TS systems

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Abstract—**This paper considers the design of observers for periodic Takagi-Sugeno fuzzy models. The observer employed is also a periodic one. To develop the design conditions, a switching Lyapunov function defined at the time instants when the subsystems switch is used. Using the developed conditions we are able to design observers for TS models where the local models or even the subsystems are unstable. The application of the conditions is illustrated on numerical examples.**

I. Introduction

Takagi-Sugeno (TS) fuzzy systems [1] are convex combinations of local linear models, and are able to exactly represent a large class of nonlinear systems [2].

For the analysis and design of TS models the direct Lyapunov approach has been used. Stability conditions have been derived using quadratic Lyapunov functions [3]–[5], piecewise continuous Lyapunov functions [6], [7], and nonquadratic Lyapunov functions [8]–[10] and have been in general formulated as linear matrix inequalities (LMIs).

For discrete-time TS models, non-quadratic Lyapunov functions have shown a real improvement of the design conditions [8], [11]–[13]. It has been proven that the solutions obtained by non-quadratic Lyapunov functions include and extend the set of solutions obtained using the quadratic framework.

Non-quadratic Lyapunov functions have been extended to double-sum Lyapunov functions in [11] and later on to polynomial Lyapunov functions in [14]–[16]. A different type of improvement in the discrete case has been developed in [9], conditions being obtained by replacing the classical one sample variation of the Lyapunov function by its variation over several samples (α-sample variation).

For a TS fuzzy model, well-established methods and algorithms can be used to design observers that estimate unmeasurable states. Several types of such observers have been developed for TS fuzzy systems. In general, the design methods for observers also lead to an LMI feasibility problem.

This paper deals with a particular class of nonlinear models with periodic parameters. This kind of models can be found in numerous domains such as automotive, aeronautic, and aerospace or also computer control of industrial process [17]–[20]. These systems can be represented by periodically switching models. In the last decade, such systems have been investigated mainly in the continuous case where the stability is based on the use of a quadratic Lyapunov function [21]–[24] or a piecewise one [25], [26]. Although results are available for discrete-time linear switching systems [27], for discrete-time TS models, few results exist [28], [29].

In this paper, we propose a switching TS observer for periodic TS models. To develop the design conditions, we use a non-quadratic Lyapunov function. This Lyapunov function is useful for designing observers for a periodic TS system having non-observable local models.

The structure of the paper is as follows. Section II presents the notations used in this paper and a motivating example. The proposed conditions are developed and extended for α-sample variation in Section III. Finally, the observer design is discussed and illustrated on a numerical example in Section IV.

II. Preliminaries

A. Background

In this paper we consider observer design for discrete-time periodic TS systems. For this, we consider subsystems of the form

$$\begin{aligned}\boldsymbol{x}(k+1) &= \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k))(A_{j,i}\boldsymbol{x}(k) + B_{j,i}\boldsymbol{u}(k))\\ \boldsymbol{y}(k) &= \sum_{i=1}^{r_j} h_{ji}(\boldsymbol{z}_j(k))C_{j,i}\boldsymbol{x}(k)\end{aligned}$$

denoted in what follows as

$$\begin{aligned}\boldsymbol{x}(k+1) &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k)\\ \boldsymbol{y}(k) &= C_{j,z}\boldsymbol{x}(k)\end{aligned}$$

where j is the number of the current subsystem, $j = 1, 2, \ldots, n_{\mathrm{s}}$, n_{s} being the number of the subsystems, $\boldsymbol{x}$ denotes the state vector, r_j is the number of rules in the jth subsystem, $\boldsymbol{z}_j$ is the scheduling vector, h_{ji}, $i = 1, 2, \ldots, r_j$ are normalized membership functions, and $A_{j,i}$, $B_{j,i}$, and $C_{j,i}$, $i = 1, 2, \ldots, r_j$, $j = 1, 2, \ldots, n_{\mathrm{s}}$, are the local models.

We consider periodic systems, i.e., the subsystems defined above are activated in a sequence $\underbrace{1, 1, \ldots, 1}_{p_1}, \underbrace{2, 2, \ldots, 2}_{p_2}, \ldots, \underbrace{n_{\mathrm{s}}, n_{\mathrm{s}}, \ldots, n_{\mathrm{s}}}_{p_{n_{\mathrm{s}}}}, \underbrace{1, 1, \ldots, 1}_{p_1}$, etc., where p_i denotes the number of samples for which the ith subsystem is active. In what follows, we will refer to p_i as the period of the ith subsystem.

0 and I denote the zero and identity matrices of appropriate dimensions, and a $(*)$ denotes the term induced by symmetry.

The subscript $z+m$ (as in $A_{1,z+m}$) stands for the scheduling vector being evaluated at the current sample plus mth instant, i.e., $\boldsymbol{z}_1(k+m)$. An underlined variable $\underline{j}$ denotes the modulo of the variable, i.e., $\underline{j} = (j \mod n_\mathrm{s}) + 1$.

In what follows, we will make use of the following results:

Lemma 1. *[30] Consider a vector $\boldsymbol{x} \in \mathbb{R}^{n_x}$ and two matrices $Q = Q^T \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{m \times n_x}$ such that rank$(R) < n_x$. The two following expressions are equivalent:*

1) $\boldsymbol{x}^T Q \boldsymbol{x} < 0$, $\boldsymbol{x} \in \{\boldsymbol{x} \in \mathbb{R}^{n_x}, \boldsymbol{x} \neq 0, R\boldsymbol{x} = 0\}$
2) $\exists M \in \mathbb{R}^{m \times n_x}$ *such that* $Q + MR + R^T M^T < 0$

Observer and controller design for TS models often lead to double-sum negativity problems of the form

$$\boldsymbol{x}^T \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(\boldsymbol{z}(k)) h_j(\boldsymbol{z}(k)) \Gamma_{ij} \boldsymbol{x} < 0 \tag{1}$$

where Γ_{ij}, $i, j = 1, 2, \ldots, r$ are matrices of appropriate dimensions.

Lemma 2. *[31] The double-sum* (1) *is negative, if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i < j \end{aligned}$$

Lemma 3. *[32] The double-sum* (1) *is negative, if*

$$\begin{aligned} &\Gamma_{ii} < 0 \\ &\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i, j = 1, 2, \ldots, r, \; i \neq j \end{aligned}$$

Property 1. *(Congruence) Given a matrix $P = P^T$ and a full column rank matrix Q it holds that*

$$P > 0 \;\Rightarrow\; QPQ^T > 0$$

Property 2. *[33](Schur complement) Consider a matrix $M = M^T = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$, with M_{11} and M_{22} being square matrices. Then*

$$M < 0 \Leftrightarrow \begin{cases} M_{11} < 0 \\ M_{22} - M_{12}^T M_{11}^{-1} M_{12} < 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases}$$

B. A motivating example

In the literature, one of the main assumptions on switching systems is that the switching can occur at any time, between any two subsystem. However, for periodic systems, the extra knowledge of when and between which subsystems the switching will occur can lead to more relaxed conditions. Consider for instance the switching TS system, composed of two subsystems, as follows. The first subsystem is a TS one, with local matrices

$$A_{11} = \begin{pmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} -0.5 & 0.2 \\ 0.4 & 0.9 \end{pmatrix}$$

while the second one is a linear one, with the state matrix given by

$$A_2 = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & a \end{pmatrix}$$

where a is a real-valued parameter, $a \in [-2, 2]$. Using a common quadratic Lyapunov function, one is not able to prove stability of the switching system for any a. Even using a nonquadratic, switching Lyapunov function, stability cannot be proven for any $a \in [-2, 2]$. However, if we know that the system switches from one subsystem to the other at every time instant, the stability of the switching system can be proven [34] for $a \in [-1.2, 1.1]$. Consequently, by using the knowledge of when and how a periodic system switches, can significantly relax the stability conditions.

Observer design condition are in general obtained from stability conditions of the estimation error dynamics. Therefore, more relaxed stability conditions in general lead to more relaxed observer design conditions. In what follows, we extend the results presented in [34] to observer design.

III. Observer design

In this section, consider the observer design problem for the periodic TS model

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{j,z}\boldsymbol{x}(k) + B_{j,z}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C_{j,z}\boldsymbol{x}(k) \end{aligned} \tag{2}$$

with n_s subsystems, each subsystem j, $j = 1, 2, \ldots, n_\mathrm{s}$, being active for p_j time samples. The observer we use is of the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{j,z}\widehat{\boldsymbol{x}}(k) + B_{j,z}\boldsymbol{u}(k) + M_{j,z}^{-1} L_{j,z}(\boldsymbol{y} - \widehat{\boldsymbol{y}}) \\ \widehat{\boldsymbol{y}}(k) &= C_{j,z}\widehat{\boldsymbol{x}}(k) \end{aligned} \tag{3}$$

that is also periodic, with the same periods as (2). For this paper, we assume that the scheduling variables do not depend on the states that have to be estimated, and consequently they can be used in the observer.

The estimation error is given by

$$\boldsymbol{e}(k+1) = (A_{j,z} - M_{j,z}^{-1} L_{j,z} C_{j,z})\boldsymbol{e}(k) \tag{4}$$

and the design conditions are equivalent to finding $M_{j,i}$ and $L_{j,i}$, $j = 1, 2, \ldots, n_\mathrm{s}$, $i = 1, 2, \ldots, r_j$ so that (4) is asymptotically stable. Note that the estimation error (4) is also a periodic TS system.

A. Design conditions

First, we consider observer design such that (4) is asymptotically stable. The results will be extended for α-sample variation, similar to [9] in the next section.

Consider the observer (3) for the periodic TS system (2), composed of n_s subsystems, with each subsystem i being active for p_i samples, $i = 1, 2, \ldots, n_\mathrm{s}$. Then, the following results can be stated.

Theorem 1. *The estimation error* (4) *is asymptotically stable, if there exist $P_{j,i} = P_{j,i}^T > 0$, $M_{j,i}$, $L_{j,i}$ $j = 1, 2, \ldots, n_\mathrm{s}$, $i =$*

1, 2, ..., r_j, *such that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{0,a} & \Omega_{0,b} & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{p_{\underline{j+1}},a} & \begin{pmatrix} \Omega_{p_{\underline{j+1}},b} \\ +P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \end{pmatrix} < 0 \tag{5}$$

where

$$\Omega_{l,a} = M_{\underline{j+1},z+l}A_{\underline{j+1},z+l} - L_{\underline{j+1},z+l}C_{\underline{j+1},z+l}$$
$$\Omega_{l,b} = -M_{\underline{j+1},z+l} + (*)$$

for $l = 0, \dots, p_{\underline{j+1}} - 1$, *where* $\underline{j}$ *denotes the modulo of* j.

Remark: Note that $\underline{j+1}$ is used because due to the periodicity the $n_\mathrm{s} + i$th subsystem is in fact the ith one.

Proof: Consider the following switching Lyapunov function, similar to the one used by [27], but *defined only in the instants when a switching takes place* in the error dynamics:

$$V(\boldsymbol{e}(k)) = \boldsymbol{e}(k)^T P_{j,z}\boldsymbol{e}(k)$$

for $j = 1, 2, \dots, n_\mathrm{s}$, if the active subsystem before the kth time instant was j.

The difference in the Lyapunov function is

$$V(\boldsymbol{e}(k + p_{\underline{j+1}})) - V(\boldsymbol{e}(k)) =$$
$$\begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix}^T \begin{pmatrix} -P_{j,z} & 0 \\ 0 & P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix}$$

The error dynamics during the $p_{\underline{j+1}}$ samples are

$$\Upsilon_{j+1} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+1) \\ \vdots \\ \boldsymbol{e}(k + p_{\underline{j+1}}) \end{pmatrix} = 0$$

with

$$\Upsilon_{j+1} = \begin{pmatrix} G_{j+1,0} & -I & \dots & 0 & 0 \\ 0 & G_{j+1,1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & G_{j+1,p_{\underline{j+1}}-1} & -I \end{pmatrix}$$

with $G_{j+1,l} = A_{\underline{j+1},z+l} - M^{-1}_{\underline{j+1},z+l} L_{\underline{j+1},z+l} C_{\underline{j+1},z+l}$, $l = 0, 1, \dots, p_{\underline{j+1}} - 1$.

Using Lemma 1, the difference in the Lyapunov function is negative definite, if there exists M such that

$$\begin{pmatrix} -P_{j,z} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & P_{\underline{j+1},p_{\underline{j+1}}} \end{pmatrix} + M\Upsilon_{j+1} + (*) < 0$$

Choosing

$$M = \begin{pmatrix} 0 & 0 & \dots & 0 \\ M_{\underline{j+1},z} & 0 & \dots & 0 \\ 0 & M_{\underline{j+1},z+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & M_{\underline{j+1},z+p_{\underline{j+1}}-1} \end{pmatrix}$$

leads directly to

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{0,a} & \Omega_{0,b} & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{p_{\underline{j+1}},a} & \begin{pmatrix} \Omega_{p_{\underline{j+1}},b} \\ +P_{\underline{j+1},z+p_{\underline{j+1}}} \end{pmatrix} \end{pmatrix} < 0 \tag{6}$$

with

$$\Omega_{l,a} = M_{\underline{j+1},z+l}A_{\underline{j+1},z+l} - L_{\underline{j+1},z+l}C_{\underline{j+1},z+l}$$
$$\Omega_{l,b} = -M_{\underline{j+1},z+l} + (*)$$

for $l = 0, \dots, p_{\underline{j+1}} - 1$.

B. α-sample variation

In what follows, we extend the result above using an α-sample variation [9] of the Lyapunov function. Then, the following conditions can be stated:

Theorem 2. *The periodic TS system* (4) *with periods* $p_1, p_2, \dots, p_{n_\mathrm{s}}$ *is asymptotically stable, if there exist* $P_{ji} = P_{ji}^T > 0$, M_{ji}, L_{ji}, $j = 1, 2, \dots, n_\mathrm{s}$, $i = 1, 2, \dots, r_j$, $l = 1, 2, \dots, \alpha$, *such that the following conditions are satisfied:*

$$\begin{pmatrix} -P_{j,z} & (*) & \dots & (*) & (*) \\ \Omega_{j+1,0} & \bar{\Omega}_{j+1,0} & \dots & (*) & (*) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \Omega_{j+\alpha,p_{\underline{j+\alpha}}-1} & \begin{pmatrix} \bar{\Omega}_{j+\alpha,t-1} \\ +P_{\underline{j+\alpha},z+t} \end{pmatrix} \end{pmatrix} < 0 \tag{7}$$

where $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$, *and*

$$\Omega_{j+i,l} = M_{\underline{j+i},z+l}A_{\underline{j+i},z+l} - L_{\underline{j+i},z+l}C_{\underline{j+i},z+l}$$
$$\bar{\Omega}_{j+i,l} = -M_{\underline{j+i},z+l} + (*)$$

for $l = 0, \dots, t-1$, $i = 1, 2, \dots, \alpha$.

Proof: Similarly to Theorem 1, consider the switching Lyapunov function defined only in the instants when a switching takes place in the error dynamics:

$$V(\boldsymbol{x}(k)) = \boldsymbol{e}(k)^T P_{j,z}\boldsymbol{e}(k)$$

for $j = 1, 2, \dots, n_\mathrm{s}$, if the active subsystem was j.

Since the Lyapunov function is only defined in the switching instants, the α-difference in the Lyapunov function corresponds to α consecutive switches in the system. Consequently, the α-difference *in the Lyapunov function* is

$$V(\boldsymbol{e}(k+t)) - V(\boldsymbol{e}(k)) =$$
$$\begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+t) \end{pmatrix}^T \begin{pmatrix} -P_{j,z} & 0 \\ 0 & P_{\underline{j+\alpha},z+t} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}(k) \\ \boldsymbol{e}(k+t) \end{pmatrix}$$

where $t = \sum_{i=1}^{\alpha} p_{\underline{j+i}}$.

The error dynamics during the t samples corresponding to the α switches in the system are

$$\Gamma_{j+1:j+\alpha}\begin{pmatrix}\boldsymbol{e}(k)\\ \boldsymbol{e}(k+1)\\ \vdots \\ \boldsymbol{e}(k+t)\end{pmatrix}=0$$

with

$$\Gamma_{j+1:j+\alpha}=\begin{pmatrix}G_{\underline{j+1},z} & -I & \dots & 0 & 0\\ 0 & G_{\underline{j+1},z+1} & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & G_{\underline{j+\alpha},z+t-1} & -I\end{pmatrix}$$

with $G_{\underline{j+i},z+l}=A_{\underline{j+i},z+l}-M^{-1}_{\underline{j+i},z+l}L_{\underline{j+i},z+l}C_{\underline{j+i},z+l}$, $i=1,\,2,\,\dots,\,\alpha$, $l=1,\,2,\,\dots,\,t-1$.

Using Lemma 1, the difference in the Lyapunov function is negative definite, if there exists M such that

$$\begin{pmatrix}-P_{j,z} & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & P_{\underline{j+1},z+t}\end{pmatrix}+M\Gamma_{j+1:j+\alpha}+(*)<0$$

Choosing

$$M=\begin{pmatrix}0 & 0 & \dots & 0\\ M_{\underline{j+1},z} & 0 & \dots & 0\\ 0 & M_{\underline{j+1},z+1} & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & M_{\underline{j+\alpha},z+t-1}\end{pmatrix}$$

leads directly to (7). □

IV. Discussion

First, let us discuss how exactly the conditions derived in Section III-A are applied. For simplicity, consider a switching TS model consisting of two subsystems, i.e., we have:

$$\begin{aligned}\boldsymbol{x}(k+1)&=\begin{cases}\sum_{i=1}^{r_1}h_{1i}(\boldsymbol{z}_1(k))A_{1i}\boldsymbol{x}(k)\\ \sum_{i=1}^{r_2}h_{2i}(\boldsymbol{z}_2(k))A_{2i}\boldsymbol{x}(k)\end{cases}\\ \boldsymbol{y}(k)&=\begin{cases}\sum_{i=1}^{r_1}h_{1i}(\boldsymbol{z}_1(k))C_{1}\boldsymbol{x}(k)\\ \sum_{i=1}^{r_2}h_{2i}(\boldsymbol{z}_2(k))C_{2}\boldsymbol{x}(k)\end{cases}\end{aligned}\tag{8}$$

Assume that the period of the first subsystem is 2, and the period of the second subsystem is 1, i.e., $p_1=2$ and $p_2=1$. The switching in the system and in the Lyapunov function are depicted in Figure 1. As can be seen, the Lyapunov function (with matrices P_1 and P_2) is defined only in the moments when there is a switching in the system: from $A_{1,z}$ to $A_{2,z}$ or from A_{2z} to A_{1z}, respectively. A 1-sample variation of the Lyapunov function corresponds to the difference between two consecutive values of the Lyapunov function. A 2-sample variation corresponds to the difference after 2 samples of the Lyapunov function, etc.

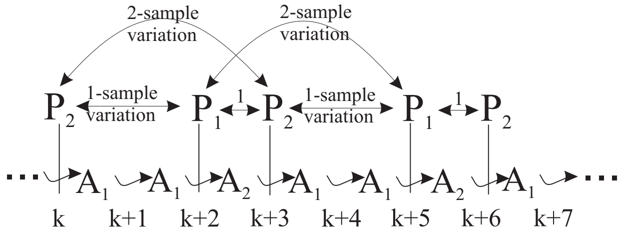

Fig. 1. Switches in the system and in the Lyapunov function.

For system (8), the conditions of Theorem 1 correspond to *there exist* $P_{j,i}=P_{j,i}^T>0$, $M_{j,i}$, $L_{j,i}$, $j=1,\,2$, $i=1,\,2,\,\dots,\,r_j$, *so that the following conditions are satisfied:*

$$\begin{pmatrix}-P_{1,z} & (*)\\ \Omega_{2,0} & -M_{2,z}+(*)+P_{2,z+1}\end{pmatrix}<0$$

$$\begin{pmatrix}-P_{2,z} & (*) & (*)\\ \Omega_{1,0} & -M_{1,z}+(*) & (*)\\ 0 & \Omega_{1,1} & -M_{1,z+1}+(*)+P_{1,z+2}\end{pmatrix}<0\tag{9}$$

with $\Omega_{i,l}=M_{i,z+l}A_{i,z+l}-L_{i,z+l}C_{i,z+l}$, $i=1,\,2$, $l=0,\,1$. Relaxed LMI conditions can be formulated using Lemmas 2 and 3, e.g.,

Corollary 1. *The system* (8) *is asymptotically stable if there exist* $P_{j,i}=P_{j,i}^T>0$, $M_{j,i}$, $j,i=1,\,2$, *so that*

$$\begin{aligned}&\Gamma^1_{i_1i_2i_3i_4i_5}+\Gamma^1_{i_2i_1i_3i_4i_5}+\Gamma^1_{i_1i_2i_4i_3i_5}+\Gamma^1_{i_2i_1i_4i_3i_5}<0\\ &\Gamma^2_{i_1i_2i_3}+\Gamma^2_{i_2i_1i_3}<0\end{aligned}$$

$i_1,i_2,i_3,i_4,i_5=1,\,2$, $i\le j$, $m\le n$, *where*

$$\Gamma^1_{i_1i_2i_3i_4i_5}=\begin{pmatrix}-P_{2,i_1} & (*) & (*)\\ \Omega_{1,i_1,i_2} & -M_{1,i_1}+(*) & (*)\\ 0 & \Omega_{1,i_3,i_4} & \begin{pmatrix}-M_{1,i_3}+(*)\\ +P_{1,i_5}\end{pmatrix}\end{pmatrix}$$

$$\Gamma^2_{i_1i_2i_3}=\begin{pmatrix}-P_{1,i_1} & (*)\\ \Omega_{2,i_1,i_2} & -M_{2,i_1}+(*)+P_{2,i_3}\end{pmatrix}$$

where $\Omega_{l+1,i_1,i_2}=M_{\underline{l+1},i_1}A_{\underline{l+1},i_2}-L_{\underline{l+1},i_1}C_{\underline{l+1},i_2}$, $l=1,\,2$, $i_1,i_2=1,\,2,\,\dots,\,r_l$.

Let us now consider a 2-sample variation of the Lyapunov function. The conditions of Theorem 2 become *there exist* $P_{j,i}=P_{j,i}^T>0$, $M_{j,i}$, $L_{j,i}$, $j,l=1,\,2$, $i=1,\,2,\,\dots,\,r_j$, *so that the following conditions are satisfied:*

$$\begin{pmatrix}-P_{1,z} & (*) & (*) & (*)\\ \Omega_{2,0} & \bar{\Omega}_{2,0} & (*) & (*)\\ 0 & \Omega_{2,1} & \bar{\Omega}_{2,1} & (*)\\ 0 & 0 & \Omega_{1,2} & \bar{\Omega}_{1,2}+P_{1,z+3}\end{pmatrix}<0$$

$$\begin{pmatrix}-P_{2,z} & (*) & (*) & (*)\\ \Omega_{1,0} & \bar{\Omega}_{1,0} & (*) & (*)\\ 0 & \Omega_{1,1} & \bar{\Omega}_{1,1} & (*)\\ 0 & 0 & \Omega_{2,2} & \bar{\Omega}_{2,2}+P_{2,z+3}\end{pmatrix}<0$$

with $\Omega_{i,l}=M_{i,z+l}A_{i,z+l}-L_{i,z+l}C_{i,z+l}$, $\bar{\Omega}_{i,l}=-M_{i,z+l}+(*)$, $i=1,\,2$, $l=0,\,1$.

Similarly to the 1-sample variation, relaxed LMI conditions can be formulated using Lemmas 2 and 3.

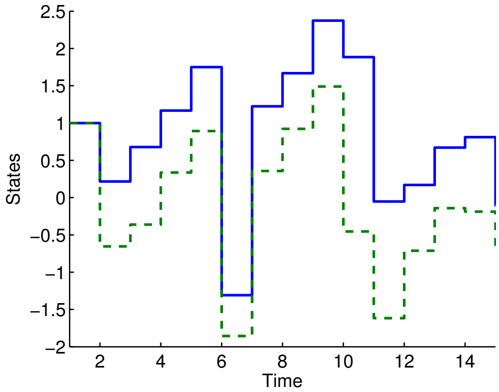

Fig. 2. A trajectory of the states of the switching system.

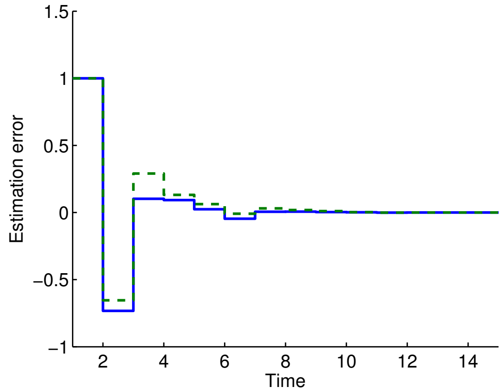

Fig. 3. Estimation error using the designed observer.

Note that the conditions do not require that the the local matrices of the TS system are either stable or observable. We illustrate this on the following example.

Example 1. *Consider the switching fuzzy system with two subsystems, each having period 2, i.e.,* $p_1 = p_2 = 2$ *as follows:*

$$\boldsymbol{x}(k+1) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))A_{1i}\boldsymbol{x}(k) + Bu \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))A_{2i}\boldsymbol{x}(k) + Bu \end{cases}$$

$$\boldsymbol{y}(k) = \begin{cases} \sum_{i=1}^{2} h_{1i}(\boldsymbol{z}_1(k))C_1\boldsymbol{x}(k) \\ \sum_{i=1}^{2} h_{2i}(\boldsymbol{z}_2(k))C_2\boldsymbol{x}(k) \end{cases}$$

with

$$A_{11} = \begin{pmatrix} 0.80 & 0.22 \\ -0.09 & 0.32 \end{pmatrix} \quad A_{12} = \begin{pmatrix} -0.82 & -0.44 \\ -1.25 & 0.33 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} 0.44 & 0.46 \\ 0.93 & 0.41 \end{pmatrix} \quad A_{22} = \begin{pmatrix} 0.84 & 0.20 \\ 0.52 & 0.67 \end{pmatrix}$$
$$C_{11} = C_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad C_{21} = C_{22} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 0 \end{pmatrix}^T$$

The local models A_{11} *and* A_{12} *are not observable, since the measurement matrices* C_{11} *and* C_{12} *are zero. Moreover,* A_{12} *is unstable, its eigenvalues being* $\begin{pmatrix} -1.1834 & 0.6934 \end{pmatrix}$*. The membership functions are as follows.* h_{11} *is randomly generated*[1] *in* $[0, 1]$*,* $h_{12} = 1 - h_{11}$ *and* $h_{21} = cos(x_1)^2$*,* $h_{22} = 1 - h_{21}$*.*

A trajectory of the states of the switching system, for the initial state $\begin{pmatrix} 1 & 1 \end{pmatrix}^T$ *and a randomly generated input is illustrated in Figure 2.*

Due to the unobservable and unstable local models, for this switching system it is not possible to design an observer using either quadratic or nonquadratic Lyapunov functions, that are common for both subsystems. Due to the unobservability and unstability of local models, the LMIs available in the literature for observer design are unfeasible.

However, using the conditions of Theorem 1 we obtain a solution. The conditions used are those in (7)*. Solving them*[2] *using the relaxation of [31], we obtain*[3]

$$P_{11} = \begin{pmatrix} 0.55 & -0.21 \\ -0.21 & 0.54 \end{pmatrix} \quad P_{12} = \begin{pmatrix} 0.55 & -0.22 \\ -0.22 & 0.56 \end{pmatrix}$$
$$M_{11} = \begin{pmatrix} 0.74 & -0.06 \\ -0.14 & 0.65 \end{pmatrix} \quad M_{12} = \begin{pmatrix} 0.69 & -0.21 \\ -0.14 & 0.58 \end{pmatrix}$$
$$L_{11} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T \quad L_{12} = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$$
$$P_{21} = \begin{pmatrix} 1.04 & -0.08 \\ -0.08 & 0.71 \end{pmatrix} \quad P_{22} = \begin{pmatrix} 1.17 & -0.08 \\ -0.08 & 0.76 \end{pmatrix}$$
$$M_{21} = \begin{pmatrix} 0.90 & -0.16 \\ 0.03 & 0.76 \end{pmatrix} \quad M_{22} = \begin{pmatrix} 0.90 & 0.05 \\ -0.17 & 0.73 \end{pmatrix}$$
$$L_{21} = \begin{pmatrix} 0.36 & 0.84 \end{pmatrix} \quad L_{22} = \begin{pmatrix} 0.87 & 0.40 \end{pmatrix}$$

A trajectory of the estimation error, with the estimated initial state being $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$ *is presented in Figure 3. As expected, the error converges to zero.*

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[1] We use a random membership function because the first subsystem is not observable.

[2] For solving LMIs, we used the *feasp* function in Matlab.

[3] All values are truncated to two decimal places.

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